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MATHEMATICAL GAZETTE

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MATHEMATICAL MODELS AND CONSTRUCTIONS.*

BY A. P. ROLLETT.

I HAD some difficulty in choosing a title for this paper. I might have called it "A Plea for More Practical Mathematics", but there would have been a doubt as to whether "more" qualified "practical" or "mathematics". And I am not very fond of the word "practical". De Morgan's remark that the self-styled practical man could always be relied on to produce some false logic is too often true. "Practical" is invariably used as the antithesis of "theoretical". The boy often regards theoretical chemistry as an unnecessary interruption of the practical work; he does not always regard them as complementary, like bread and butter.

Now mathematics is often regarded as the bread-and-butter of science, to use the metaphor in another way. But the butter is so often omitted, and the result is indigestion. The pupil becomes allergic to mathematics: he cannot live by bread alone. I want to suggest a few ways of buttering the bread, by the introduction of more constructional work and by keeping the teaching in closer touch with real things. I will not say "in closer touch with reality": reality means examinations and their attendant syllabuses to many people. By "models and constructions" I mean any objects or drawings, made by the pupil or his fellows, which help to illustrate the matter in hand. I do not wish to turn every mathematical class room into a chaos of paper, string and glue, nor to drive every teacher frantic with the clumsy efforts of the unhandy. All I suggest is that the opportunities for model making or drawing should be there, the materials ready to hand, and that the work should be done without fuss, by those able or willing to undertake it. The qualities of neatness, dexterity, patience and skill are required in varying proportions, and there is sooner or later a job for everyone.

I am well aware that many professional mathematicians have scorned the use of models and even of figures. There are two celebrated examples: Lagrange's "Point d'images dans cet ouvrage" in the *Mécanique Analytique*, and Laplace's frequent and disconcerting "Il est aisé à voir" in the *Mécanique*

* A paper read to the General Meeting of the Mathematical Association, April 6, 1945. Models and diagrams were on view.

Céleste. However, I am content to follow Clerk Maxwell, and I should like to quote a few passages from his introductory lecture on Experimental Physics as the first Cavendish Professor in 1871 which have some bearing on my theme.

"There is no more powerful method for introducing knowledge into the mind than that of presenting it in as many different ways as we can. When the ideas, after entering through different gateways, effect a junction in the citadel of the mind, the position they occupy becomes impregnable.

"It is therefore natural to expect that the knowledge of physical science obtained by the combined use of mathematical analysis and experimental research will be of a more solid, available, and enduring kind than that possessed by the mere mathematician or the mere experimenter. But what will be the effect on the University, if men, pursuing that course of reading which has produced so many distinguished Wranglers, turn aside to work experiments? Will not their attendance at the Laboratory count not merely as time withdrawn from their more legitimate studies but as the introduction of a disturbing element, tainting their mathematical conceptions with material imagery, and sapping their faith in the formulae of the textbooks?"

He then speaks of "the difficulty of recognising, among the concrete objects before us, the abstract relation which we have learned from books, and the distracting pain of wrenching the mind away from the symbols to the objects and from the objects back to the symbols. This, however, is the price we have to pay for new ideas.

"But when we have overcome these difficulties and successfully bridged the gulf between the abstract and the concrete, it is not a mere piece of knowledge that we have obtained: we have acquired the rudiment of a permanent mental endowment. When, by a repetition of efforts of this kind we have more fully developed the scientific faculty, the exercise of this faculty in detecting scientific principles in nature, and in directing practice by theory, is no longer irksome but becomes an unfailing source of enjoyment, to which we return so often that at last even our careless thoughts begin to run in a scientific channel."

I feel that this is not irrelevant in spite of the fact that Clerk Maxwell is talking about physics. As someone has said: "Analogy is no argument, though it may suggest the existence of one."

I should like to quote one more paragraph, from the same lecture, which emphasises an important truth.

"The amount of a man's energy is limited, but its most effective use is to be obtained by a proper distribution between mathematical study and experiment and not in an entire concentration on the mathematics. A great part of our fatigue often arises, not from those mental efforts by which we obtain the mastery of the subject, but from those which are spent in recalling our wandering thoughts. and these efforts of attention would be much less fatiguing if the disturbing force of mental distraction could be removed. This is the reason why a man whose soul is in his work always makes more progress than one whose aim is something not immediately connected with his occupation. In the latter case the very motive of which he makes use to stimulate his flagging powers becomes the means of distracting his mind from the work before him. There may be some mathematicians who pursue the study entirely for its own sake. Most men, however, think that the chief use of mathematics is found in the interpretation of nature. Now a man who studies a piece of mathematics in order to understand some natural phenomenon which he has seen, or to calculate the best arrangement of some experiment which he means to make, is likely to meet with far less distraction of mind than if his sole aim had been to sharpen his mind for the successful practice of the Law or to obtain a high place in the Tripos."

Clerk Maxwell said he had "known men who, when they were at school, never could see the good of mathematics, but who, when in after life they made this discovery, not only became eminent as scientific engineers, but made considerable progress in the study of abstract mathematics."

We, too, know boys at school who do not see the good of mathematics, and who are tired with or without the effort of recalling their wandering thoughts. It is a widely-held belief that mathematics—especially pure geometry—is a sound logical training. That belief is not shared by those who have examined school certificate geometry scripts. My own view is that no study is of the slightest value in developing the logical or any other faculty unless the interest is there. There will always be those who will never be logical—or, to put it more kindly, whose hearts will always rule their heads—but I am convinced that mathematics can be appreciated and understood by a wider circle than is usual. But when a boy says, "What is the good of this?" he is telling you that he is not interested. He does not say, "What is the good of rugger?" and if he plays rugger it isn't because he knows it does him good. The teacher must take the hint, and see that at the right times the boy has something to make or do in which there is a definite aim, on which he will concentrate willingly. Concentration, I am sure, is a faculty which is capable of transference to other spheres, once the habit has been acquired.

A friend once said to me: "If only I knew more about engines and things I could make my teaching so much more interesting." I said I was sure that was a fallacy. I have so often found a division much more enthusiastic over the calculation of π to a few decimal places than in the discussion of the theory of flight. When we try to be topical we often distract: the boy is thinking of the fun of flying as much as about lift and drag, but in calculating π he knows what he is after, and feels he is taking part in a chase of some kind. And the making of a mathematical model or the plotting of a difficult graph is undertaken by most boys with as much readiness as the construction of a model aircraft. There are boys who get along very well without help from models or careful drawings, but it is a pity to debar the many not gifted with much visual imagination from making the progress in mathematics of which they are capable. The boy who develops rather slowly (but none the less surely) usually gains greatly in confidence if he can literally see what he is about. But the interest must be created in the boy's mind by the teacher, and not imposed from outside. As J. W. A. Young says in his *Teaching of Mathematics*: "A boy who finds a problem dry and meaningless will not suddenly be overwhelmed with interest when shown a neat way of solving it with squared paper." The boy must *want* to work the problem.

So far I have referred vaguely to models and drawings without indicating the scope or the nature of the constructional work involved. I am of course only concerned with the usual mathematical course from twelve-year-olds to those entering for University Scholarships. Naturally, when one is discussing, say, a pyramid, it is convenient to have several made to assist understanding, and if coaxal circles are on the bill it is helpful to have a really nicely drawn system to exhibit. But I believe very strongly in seizing the slightest opportunity for a digression in any direction, especially if the element of surprise can be introduced. G. H. Hardy has given "unexpectedness, inevitability and economy" among the qualities of a mathematical theorem. We often in our teaching dwell too long on one topic, perhaps confusing boredom with lack of comprehension. That is the time to go right off the track, to bring in new material. It doesn't matter if this involves curves not mentioned except in so-called higher mathematics, and it doesn't matter if only a short time is spent on it, but some curiosity will have been aroused and some subconscious cerebration set in motion.

It seems absurd that in school mathematics the only loci officially recognised for elementary examination purposes should be the straight line and circle. A boy has only to plot three reasonably safe points and he knows which it is. In the same way nearly all the graphs in elementary algebra are parabolas. One of my earlier pupils used to murmur loudly and derisively " same old curve " whenever the unfortunate parabola appeared, and when we began mechanics and it appeared again he swooned. This stung me into broadening my syllabus. I was reminded of Voltaire's taunt about our having a hundred religions and only one sauce.

A boy ought to know several more curves, not vaguely as appearing among the optional questions at the end of the exercises on loci, but as part of his equipment. He ought to know several ways of drawing a parabola, ellipse and hyperbola, and should know the shape of the cycloid, cardioid, lemniscate, and so on. He ought to be on the look-out for mathematical forms in art, nature, or engineering. He ought to regard it as natural to draw a mathematical curve in designing a boat or a piece of furniture : he should regard it as bad form in both senses to make a tea-tray bounded by four circular arcs instead of a true ellipse.

In 1901 the British Association discussed the teaching of mathematics, or rather, they discussed Professor Perry's ideas on the subject. Professor Perry gave a list of eight " obvious forms of usefulness in the study of mathematics ". He put first : " In producing the higher emotions and giving mental pleasure, hitherto neglected in teaching almost all boys." And he was an undoubtedly practical man.

Oliver Heaviside, whose name is now known to a wide circle of radio enthusiasts, sent a written comment on Prof. Perry's address (written because I believe Heaviside was a recluse of the most curmudgeonly order). In it he says : " Boys are usually exceedingly stupid in anything requiring concentrated reasoning. It is not in the nature of their soft brains that they should take kindly to Euclid and other stuff of that logic-chopping kind. But they usually possess another sort of mental ability—namely, the ready acquisition of new facts and ideas—and that is what should be taken advantage of. They have also the power of learning to work processes long before their brains have acquired the power of understanding (more or less) the logic of what they are doing. Now the prevalent idea of mathematical works is that you must understand the reason why first, before you proceed to practice. This is fudge and fiddlesticks."

There is no doubt that many boys prefer to know how to work a process first ; then, when they are thoroughly familiar with it, they are ready and even anxious to know the reason why. And reading another B.A. discussion (on the Teaching of Mechanics), at the 1905 meeting in Johannesburg, I came across this from Prof. Forsyth : " What is wanted in the first instance is to accustom students to deal with things the properties of which they will have an opportunity of argument upon later. Do not begin with definitions—begin with the things themselves."

I feel that so far I have produced a very uninspiring overture with two elusive themes which perhaps I ought to restate. I want to try to put a little more life into elementary mathematics firstly by contriving that the pupil shall at times handle or at least have frequent glimpses of the stock-in-trade of more advanced work, and secondly by keeping the aesthetic side in view as much as possible. And I believe that both objects can be attained by a judicious use of work involving the hand and eye as well as the mind. To put it paradoxically, a boy can be interested in the abstract if only it is set before him in a sufficiently concrete form.

No one would expect a fourth-form boy to know anything about ruled

surfaces, but I have kept several models on the mantelpiece in my room for years. A hyperboloid is easily made by threading string through saw-cuts made in the ends of an empty wooden spool of the kind used for electric wire. It causes a good deal of interest, for the boy is puzzled to see a curved *surface* formed of straight strings, even if he is familiar with plane envelopes. But his imagination is stimulated and his eye quickened, and he later notices the phenomenon (as it is to him) exemplified in baskets and the ornamental fences protecting young trees in the park. If he is interested in lathe-work he can now understand the result of attempting to turn a cylinder when the tool moves in a straight line skew to the axis. The curious concave spindle has puzzled most workmen. And quite recently a boy showed me an article in a scientific periodical which suggested that the correct shape for a plough is part of a hyperboloid.

And the model of an ellipsoid made entirely from two interlocking sets of parallel circular sections is found very fascinating. It is collapsible in two ways, and is not at all hard to make : boys connect it with the geodetic construction of the Wellington bomber. Instructions for making it will be found in *Analytical Geometry of Three Dimensions*, by W. H. McCrea, published recently by Oliver and Boyd in their series of five-shilling mathematical texts. The inclusion of this and other models naturally prejudiced me in the author's favour, a feeling strengthened by a perusal of the rest of the work.

Models like this cause a good deal of argument, and with boys (as with politicians) argument is often followed by thought or by a desire to know more of the relevant facts. They add a little zest to the study of mathematics, and a division comes into the room with a faint though noticeable air of expectancy.

I propose now to consider the work suggested in more detail, subject by subject as far as possible.

Arithmetic does not lend itself to experimental illustration so readily as geometry, and I suppose pure computation is the nearest counterpart to the making of graphs or loci. Much valuable and interesting work can be found in the compilation of tables of integral powers, square roots, and factorials. Even logarithms can be calculated with a little team-work by fairly simple processes, and cube-roots by one of the many formulae of iteration. The periods of the reciprocals of prime numbers are quite good sport, since the number of digits may be $(p - 1)$ or any factor of it. In all these the work can be checked by the boy himself, whose normal accuracy is not so great that time willingly spent on pure computation can be considered as time wasted. It is a good plan to keep the results in a book, signed by the computer, who will then feel he has contributed a piece of mild research.

If mensuration is regarded as part of arithmetic, then simple models are often desirable, especially when the height or the volume of a pyramid is to be found. A square pyramid with one slant face absent and a piece of string as the altitude is helpful to those who don't understand the figure in the book or on the blackboard. Even if every teacher of mathematics were a good draughtsman, I believe the fact that he draws in white on a black surface and the pupil in black on a white surface causes some misunderstanding. A model showing six square pyramids forming a cube is worth making, if it is partly collapsible. For most of the simpler models glue is unnecessary, and it is certainly undesirable. In W. H. Young's *First Book of Geometry*, published by Dent in 1905, will be found the nets of many interesting solid figures which achieve rigidity by means of interlocking flaps.

It is customary to begin a course of physics with what is called " Practical Measurements ", which includes the use of such instruments as balances, calipers, verniers, and burettes. Regarded as physics it is usually found

extremely dull, but if it can be done as mathematics it is not only interesting but valuable, especially if percentage error and elementary statistical ideas are introduced. The trend of thought in physics makes the consideration of statistics and probability not only desirable but urgent, and experimental work is possible. The approximate calculation of π by the pencil and boarded floor method is worth doing. It is easily carried out as a class experiment, each member dropping a piece of matchstick or wire about $\frac{3}{4}$ inch long a hundred times or more on to paper ruled with parallel lines one inch apart. The probability that a stick of length s inches will cross one of the lines is $2s/\pi d$ if the lines are d inches apart, and the proof can be given as soon as the integration of the sine is reached. And the boy's natural instinct to use a pair of compasses as a dart can be sublimated similarly by using the collective results of aiming at a target, placed on the floor by the side of his desk, to illustrate certain statistical terms.

Algebra, with its graphs, provides many interesting things to do. Of course the simple graphs of straight lines and parabolas must be done, but I suggest that whole families of curves should be drawn. To draw one straight line is rather futile, but to draw a series of lines $y = mx + c$ with (say) m constant and c varying by regular steps, not only makes a better pattern but satisfies a boy's sense of orderly progression. Families of parabolas can be drawn, using a cardboard stencil or jig to save labour. The idea of a family of curves should not be left until differential equations are studied, and I believe strongly in the value of pattern and progression. The curves $y = x^n$ for ranges of values of n, the sequence $y = x$, $y = x(x-1)$, $y = x(x-1)(x-2)$, and so on, are also interesting. The results of individual work can be copied on to one sheet in miniature, or all drawn with the same axes according to the character of the curves. Boys are interested and amused by curves with isolated points or loops, or having curious shapes, and x^x for negative x can shake him out of his complacency. The boy who dislikes using other than integer values of x can be cured by being asked to plot $y = x^3(5 - x^2)$ from $x = -2$ to 2; the five easiest values give collinear points.

The idea of giving families of graphs sometimes leads into interesting byways. I once gave $y^2 = x^3 - 3x^2 + p$ for values -2, 0, 2, 4, 6 of p. This gives an attractive pattern, showing loop, double point and isolated point, which was the object in view. But one boy thought it looked like a contour map of part of the Alps, and suggested that a model be made to illustrate this. The opportunity of extending to three-dimensional coordinate geometry by way of this function of two variables was too good to miss, and the Alps became the surface $z = y^2 - x^3 + 3x^2$. In a week-end one boy—who had volunteered for the job—produced a model entirely on his own, made from cut-out cardboard sections fitted together on the egg-box principle. This model was very well received, and I had requests for equations which would give similar three-dimensional models, and several others were made. They involved much calculation apart from patience and perseverance.

Many interesting curves can be found in Frost's *Curve Tracing*, and although some of the equations seem forbidding they often require no more than a series of quadratics to be solved; sometimes the only way is to find where $y = mx$ cuts the curve. But solving quadratics arising in the construction of an elegant curve is lightly undertaken—there is an object in view. And with a difficult equation tabulation of the working is seen to be necessary, "continuity" acts as a check, and many useful lessons are learnt. After laborious plotting of this kind, the principles used in the more general tracing of "higher plane curves" are very cordially welcomed and appreciated. Any interesting curves should be reproduced on a larger scale and stored for future exhibition or reference. A boy gets quite a kick on seeing that the

name at the foot of a well-drawn and difficult graph is that of the school's fast bowler.

But graphs are not the only forms of practical algebra. The study of the Binomial Theorem is considerably enlivened if a coin-tossing machine can be brought into operation. The one I use requires a crew of three—a striker, an observer, and a recorder. A large number of tosses of (say) five halfpennies can be made in a short time, and there is no shortage of willing crews even out of school hours. By making the number of tosses some multiple of 2^5 the results can be compared with the coefficients in $(h+t)^5$. A frequency distribution of (say) barometer heights or lengths of carrots can be compared in shape with a column-graph of the binomial coefficients, and a Galton Quincunx is suggestive of the way in which heredity may operate. Boys will spend a long time trying to get an angle of slope that will give the desired binomial distribution.

In Geometry the field for constructional work is naturally inexhaustible, and yet how many school-books give even a hint of the possibilities? The outlook is much brighter in the technical schools, where the books used are full of interesting and decorative work, which would help the ordinary course in theoretical geometry by emphasising the practical and aesthetic sides. Surely geometric form is the basis of good design (though I would not say that for a design to be good it need only be geometrical). I should like to put in a word here in favour of the introduction of a short course of what is called Mechanical Drawing in schools. We started it at Sevenoaks because we had to fill up the time-table of a wood-work instructor, and it proved so valuable (and popular) that we have extended it to all boys. To draw plans, elevations and sections properly on a drawing-board is a great help to three-dimensional thinking, and well worth the time spent on it, which need be only an hour a week for a year or so.

At the elementary stage quite a lot of fun can be had with mixed polygonal tile-patterns, dissection puzzles, and the glueless models of cubes and their various frusta (leading to the tetrahedron and octahedron) described by W. H. Young in the book already quoted. The five regular Platonic solids can be made and discussed, and any simple concurrencies (medians, etc., of a triangle) and other interesting geometrical results easily constructed. There are so many varied loci and the idea is so much neglected that there is no excuse for not starting early. In addition to the ordinary point and line loci, there are the conchoids, the Archimedean spiral, and the many "trammel" curves. These last are the loci of points P on a line AB, where A and B travel along fixed lines or circles, and can be described mechanically. They include the locus of a point on the connecting rod of an engine. The study of envelopes should be parallel with that of loci, and there are many striking and not too difficult examples. The easiest is perhaps the parabolic envelope, made by joining $(p, 0)$ to $(0, k-p)$, the axes being not necessarily rectangular. Then there is the astroid, and the pin-and-setsquare envelopes which give the parabola and ellipse (or hyperbola) from the tangent at the vertex and the auxiliary circle respectively. But the most fascinating straight-line envelopes are the epicycloids. To draw these, 24 or 48 equidistant points are marked on the circumference of a circle, and numbered consecutively. To get $(n-1)$ cusps, join the points 1, 2, 3 ... to the points n, $2n$, $3n$ ... , and so on. A hypocycloid of $(n+1)$ cusps is obtained by joining 1, 2, ... to $-n$, $-2n$, etc. and producing the lines, but this is harder and needs more accurate drawing. Circle envelopes are well worth doing. Ignoring one or two obvious examples, an interesting case is the envelope of circles whose diameters are parallel chords of a fixed circle: the result is an ellipse and two sets of circles condensing towards its foci. A harder one for those who can draw a rectangular

hyperbola by any means is the envelope of circles passing through the origin whose centres are on this curve. The result is Bernoulli's lemniscate, and is very striking. I remember Lord Dunsany being so fascinated by it at one of our exhibitions that I expected it to figure in a future Jorkens story.

Then there are the curves of pursuit, the farmer and the pig, the dogs at the corners of an equilateral triangle each chasing the next, and the tractory. It is worth while having some examples of linkages, the pantograph, Sylvester's plagiagraph, the straight-line motions of Peaucellier, Hart and Kempe, and the nearly-straight-line motions of Watt, Roberts and Tschebycheff. These generally arouse much interest: they can be made from Woolworth curtain rail; Meccano is not so good as there is too much "slack" at the joints.

It is useful to have good diagrams drawn of the standard results of Higher Geometry, the nine-points-circle, the theorems on intersecting circles, Feuerbach's theorem, the two centroid-equilateral-triangles (and this makes a very interesting exercise on vectors), centres of similitude of three circles, and so on. Interesting figures to draw are those illustrating the properties of four lines (Wallace point, etc.) and of four concyclic points, and hardier spirits can tackle five lines.

Some years ago I was doing Simson's line with a not very bright sixth form when one of them said, "What is its envelope?" I received this with mixed feelings of satisfaction at the results of earlier work and dismay at my ignorance of the answer. The question was tactfully deferred, and in a day or two the questioner produced a satisfactory drawing and a mathematical specialist, who had been working in a corner of the room, had worked out the solution by the usual calculus method. Later I came across other solutions in back numbers of the *Mathematical Gazette*, but we got quite a lot out of the affair. It makes a nice picture with the three-cusped hypocycloid concentric with the nine-points circle, or with Morley's triangle superimposed.

There are, of course, many solid models to be made. The thirteen semi-regular (Archimedean) solids, solids of interpenetration which call for some skill with wire, and the difficult star and stellated polyhedra form a most impressive display, especially if given their full names. There was a fine show of them here last year.

But there are many models which are useful as well as ornamental, such as tetrahedra of various types, and a tetrahedron inscribed in a celluloid or wire parallelepiped is much clearer than a blackboard drawing. Theorems such as the shortest distance between two lines are much easier if a model is available.

To return to elementary geometry, I feel that every opportunity of reinforcing the work by reference to the constructional side should be taken. The theorem about the line of centres of two circles seems to be regarded as trivial by the average boy, judging from the travesty of a proof that he serves up. He would probably take it more seriously if he realised its use in the drawing of arches or even of sections of drains. And a boy ought to know how to produce parabolic, elliptic or hyperbolic arcs by methods suitable for a workshop, whether by envelope, trammel, or intersecting pencils. Some of you may remember a lecture here a few years ago by Mr. Romney Green, the mathematician, boat-builder and furniture-maker of Christchurch, Hampshire. His notepaper is headed: "Fair Hulls ensured by Geometrical Methods." Anyone who knows the method followed by the ordinary boat-builder in laying out his curves will not be surprised to find Romney Green's boats more satisfying. But I think his slogan ought to be chalked up in every school workshop.

Some of this work will obviously not appeal to girls, but in compensation there is much they can do in a sort of embroidery by using coloured threads instead of ruled lines. Curves of pursuit, Archimedean spirals, parabolic and

epicycloidal envelopes make attractive designs, and I have to thank my daughter for working a few specimens. You will find more about these in a little book written in 1906 by Mrs. Somervell with the curious title of *A Rhythmic Approach to Mathematics*.

The earlier stages of trigonometry are usually found interesting enough, and outdoor practical work is possible. It is, of course, desirable to have models of pyramids, spheres (for latitude problems), and what is known as the hillside figure. But large angles are rather dull, and so it is very helpful to do some plotting of polar curves, which gives good practice in dealing with them. For museum purposes it is worth buying good lithographed polar coordinate paper, and the curves can be plotted in inverse pairs, circle and line, cardioid and parabola, and so on, to save expense. The only practical demonstration I remember as a schoolboy was the drawing of a cycloid on the blackboard by means of a round three-legged stool, a piece of chalk being tied to the base by a string. It was spontaneous and consequently effective. Cycloids, epicycloids and the rest of the family give very good practice in the use of circular measure if their parametric equations are found. Bridges sometimes have cycloidal arches; there is the interesting fact that the two-cusped hypocycloid is a straight line; and the trochoid illustrates the old catch-question: "Which parts of a train travel in the opposite direction to the engine?" Ingenious mechanisms for drawing these curves are often made by boys with Meccano. The fact that a sine curve is the orthogonal projection of a screw thread or the handrail of a so-called "spiral" staircase ought to be known. The "ellipse" which can be drawn on the curved surface of a cylinder by means of ordinary compasses gives an interesting exercise in trigonometry. Everybody knows that when a cylinder is sawn off at an angle the section is an ellipse, but few realise that if paper is wrapped round first its edge will when unwrapped form a sine curve: shelf-paper for the mathematical larder!

In passing, it might be noted that a spherical triangle is formed by the curved edges of a suitably folded circular filter paper.

There is little doubt that the study of calculus and coordinate geometry is greatly helped when several different curves are already familiar. A boy who has come across half-a-dozen ways of drawing an ellipse is interested to see how they can all be proved to be the same kind of curve by means of their equations.

Polar and parametric coordinates will crop up in trigonometry, but it is interesting to keep in touch with other systems, especially bipolar coordinates. From quite simple relations a wide range of interesting curves can be drawn, including lines, circles, cartesian ovals (including the ellipse and hyperbola), Cassini's ovals (including the lemniscate). I should like to protest against the use of the term oval as if it applied only to a symmetrical curve, such as an ellipse. Oval means egg-shaped, and, appropriately enough, the only bird laying a nearly elliptical egg is, I believe, the cuckoo. The first piece of research undertaken by Clerk Maxwell, at the age of fourteen, was on the description of cartesian ovals by means of pins and string: he did not discover the third focus. With a little trouble it is possible to work out cases where the same oval can be drawn in two ways from two of the three foci, without undue strain on the string. This makes an interesting digression, and there is information in Williamson's old *Calculus* and in Minchin and Dale's *Mathematical Drawing*.

Talking of string, the rather neglected involute is easy to draw if nails are hammered into the parent curve and a string of suitable length and carrying a pencil is unwound from them. The involute of a circle is used in gear teeth, and is the path of a point on a see-saw when a cylindrical log is the fulcrum. The parabola and ellipse can be shown as involutes of their evolutes.

The involute of the cycloid is interesting in its connection with the cycloidal pendulum. An interesting sequence of roulettes is as follows. Draw (mechanically) the involute of a circle of radius $2c$. Roll this involute (still attached to its circle) on a straight line, and the centre of the circle traces out a parabola of latus rectum $4c$. Roll this parabola on a straight line: its focus describes a catenary of parameter c. In practice the parabola, with a hole at its focus, can be rolled in front of an actual hanging chain. But the most fascinating suites of curves in the field of the calculus are those formed from increasing numbers of terms of a Maclaurin expansion. They are not difficult if the work is decently tabulated, and the approximations to $y = \sin x$ are particularly striking. When functions of two variables are studied a solid model is most helpful.

For the greater part of my career I have taught physics as well as mathematics to the sixth form, and so can see each subject through the eyes of the other. Within the scope of this paper there are a number of topics of interest to both sides. On the elementary level there are the many simple optical illusions, surely a subject of importance to any experimenter. It is useful to know that a reflection caustic is an epicycloid (I have a little model which shows this), and that a refraction caustic is the evolute of an ellipse of eccentricity $1/\mu$. The correct shape for a refracting surface is a cartesian oval (drawn for glass with one double and one triple string). Cassini's ovals and the hyperbola occur in polarised light. Sine curves can be added to show the effect of harmonics and the phenomenon of "beats". Lissajou figures give examples of parametric equations. Magnetic lines of force and equipotential lines are bipolar loci of a somewhat tedious but not impossible kind. And the graph of damped oscillatory motion is important.

If Biology is studied in the school, then a perusal of D'Arcy Thompson's unique book, *On Growth and Form*, will convince anyone that mathematics has something to contribute. The regular and semi-regular solids, particularly the space-fillers, are important in cellular structure. The wider definition of a gnomon as a figure which, added to a figure A, produces a figure B similar to A, leads to some interesting results. The gnomon of a rectangle with sides in the ratio $\sqrt{2} : 1$ is a congruent figure (and consequently this is a suitable shape for photographic plates or paper). If the sides are in the ratio of the "golden mean", the gnomon is a square. If the process is continued, corresponding points in the successive figures lie on an equiangular spiral, an important curve in organic growth.

I don't like to make a distinction between "pure" and "applied" mathematics, so I shall not apologise for airing my views on experiments in Mechanics. I don't believe in playing at discovering principles; it rarely works honestly. I find it much better to state a principle first and afterwards verify it with the simplest possible means, preferably having the appearance of improvisation. The one exception is in the case of machines, when it is best to use the real thing. Elaborate models do not always produce the right effect: the boy says, "What, all this to prove just that?" A Physics examiner writing in the *School Science Review* has said recently that he believes half the children in the country regard Archimedes' principle as a phenomenon associated solely with a bucket and cylinder. There are several simple paradoxes which provoke a great deal of useful discussion and are consequently of real value. The reel paradox is well known and ought to be thoroughly understood, and the double-cone which runs uphill is another old one. The cylinders of equal mass and unequal moments of inertia are interesting (especially if one is filled with liquid). The principle of the servo-brake or the torque-amplifier gives a very puzzling friction-paradox, which some of you may not have seen. And much discussion about centripetal force is caused

by the rotation on a gramophone table of two test-tubes, one containing water and a piece of lead, the other water and a small cork : the cork sinks and the lead rises. And with the same gramophone a rough value of g can be found with the aid of a tin of Lyle's Golden Syrup and a pocket torch. The angular velocity of the turntable is found stroboscopically in the usual way, and a simple optical method gives the focal distance of the parabolic surface of the rotating treacle. Knowing ω and f, g is easily calculated. But quite a lot can be done in dynamics using two heavy simple pendulums and plasticine, and several statical principles can be convincingly illustrated by no more than a poker, nails, and string. For some unorthodox experiments in mechanics, see Goodwill, *Elementary Mechanics* (Oxford, 1920), and Pohl, *Mechanics and Acoustics* (Arnold).

There are many omissions from my catalogue : I have not mentioned the Möbius ribbon and the Klein bottle, angle trisectors and nomograms, or the soap film experiments. But I have tried to indicate an attitude and to some extent a policy. If I appear to suggest that most of the time should be spent on digressions, it is only because the work outlined is in fact to be spread over many years and many hands. The Mathematical Museum or Collection —whatever it is called—should not have everything in the shop window. Familiarity has its drawbacks. It is not necessary or even desirable that every piece of work should be of high quality. A poor attempt gives someone else the opportunity to benefit himself and the community by making a better one. A boy likes to feel that he is making a contribution of lasting value. And there is a virtue even in peace-time of using scrap materials. It is quite common for a school to hold a scientific conversazione : a similar show dealing with mathematics alone can be just as interesting and valuable, and it means that Mathematics becomes News.

It will, of course, be argued that no time can be found for such work. The first step towards finding time for anything is to want to do it : there is a saying among golfers : " If golf interferes with work . . ." But in schools there are many times when normal work is disturbed or falls flat, after examinations, during an epidemic, or in the days before a Speech Day or some other world-shaking event. The boy who gets to the end of the exercise too soon can be kept quiet and usefully occupied if he has a task to fall back on. But my own experience is that quite a lot of time can be spared, because the resulting greater mathematical awareness enables more of the ordinary work to be done in the remaining time. And I find that in any group of twenty boys there are always a few who are useful with tools, who can solder or draw or paint well, and they are often the slower people whose mathematical sense is quickened by the work. If I were a trained teacher I could bring in phrases like the development of spatial perception, the function concept, and so on : but I am not.

The few exhibits I have brought along are nearly all several years old, or recently repaired, as I have not had the opportunity for this kind of work since the war began. I am sorry that so many are of crude design, but I hope that other members of the Association will send up their own ideas to the *Gazette* or to some future meeting, for I know that hardly anything of what I have to show is new. But the information about models is scattered in so many books, several of which were almost unobtainable even before the war, that I felt it worth while to make a small collection, even if it is old wine in not very new bottles. Some of you may retort that it is the kind of wine offered to Alice by the March Hare.

I should like, before I conclude, to quote a few paragraphs from A. N. Whitehead's Presidential Address to this Association in 1917, towards the end of the first world war. His subject was Technical Education, and his

words seem relevant to the present time. "The early Benedictine monks", he says, "rejoiced in their labours, because they were thereby made fellow-workers with Christ. Stripped of its theological trappings, the essential idea remains, that work should be transfused by intellectual and moral vision, and thereby turned into a joy, triumphing over its weariness and its pain. The immediate need of the nation is a large supply of skilled, efficient workmen, of men with inventive genius, and of employers alert in the development of new ideas. There is only one way . . . by producing workmen, men of science, and employers who enjoy their work. . . . Is it likely that a tired, bored workman, however skilful his hands, will produce a large output of first-class work? He will limit his production and be an adept at evading inspection; he will be slow in adapting himself to new methods; he will be a focus of discontent, full of impracticable revolutionary ideas. If, with the troubled times which may be before us, you wish to increase the chance of some savage upheaval, introduce widespread technical education and ignore the Benedictine ideal. Society will then get what it deserves. Again, inventive genius requires pleasurable mental activity as a condition for its vigorous exercise. 'Necessity is the mother of invention' is a silly proverb. 'Necessity is the mother of futile dodges' is much nearer the truth. The basis of the growth of modern invention is science, and science is almost wholly the outgrowth of pleasurable intellectual curiosity."

I apologise for quoting at such length, but in view of yesterday's discussion it seemed specially apt.

How is this interest to be achieved? In a broadcast talk a few months ago Mr. Romney Green said that he does it by arranging that each of his workmen is wholly or mainly responsible for a piece of work which he knows to be good or beautiful even if it is not his own design. I am sure that for "workmen" we can read "boys", in this and in Whitehead.

In conclusion, may I remind you of the mathematics which the Mock Turtle studied?—Ambition, Distraction, Uglification and Derision. Are our own pupils so much better off? But it was at least my Ambition, at the risk of driving you to Distraction, to suggest a few ways of removing the Uglification from school mathematics. The Derision may come from my audience.

A. P. R.

CORRESPONDENCE.

MATHEMATICAL MODELS.

To the Editor of the *Mathematical Gazette*.

SIR,—It was clear to any member present on the second day of the recent Annual General Meeting of the Association that the demonstration of models and graphs which accompanied Mr. Rollett's excellent lecture excited considerable interest.

It is presumably impracticable to publish a complete catalogue with descriptions in the *Gazette* at the present time. Nevertheless, since these details may be of considerable interest to teachers, would the Teaching Committee be prepared to consider the possibility of incorporating them in a short report on this topic?

Yours etc.,

B. M. BROWN.

DIFFERENTIALS FROM A NEW VIEWPOINT.

By A. Barton.

1. *Object of the Article.*

It is not my purpose to expound a new Theory of Differentials in the sense used by Professor G. Temple in his paper to the Mathematical Association in 1936 (G. Temple, "The Rehabilitation of Differentials," *M.G.*, Vol. XX, p. 120), but merely to embark on what he called the "comparatively trivial business" of giving an interpretation to the differential notation which presupposes the definition of derivatives, and to do so from the point of view of School rather than University teaching. A study of elementary textbooks on the Calculus shows two main schools of thought. The first, and older, of these regards differentials as infinitesimals, that is to say as variables which tend simultaneously to zero and whose ultimate ratio is their only interesting property. This presents great difficulties to beginners and savours too much of the early days when the theory of limits had not been developed. It has been largely ousted by the second school, which regards the differentials of independent variables as actual increments and defines the differential of a dependent variable (in terms of them) as what I like to call a "bogus increment". It seems to me that this second interpretation is not as easy as it pretends to be, and that it suffers from serious drawbacks; and the purpose of this article is to put forward a third interpretation which so far as I know is original and which is free from the objectionable features of the other two. Of what follows, the earlier part is a criticism of the second interpretation referred to, while the later part is an attempt to show how differentials can be more easily taught, and to demonstrate the adequacy of my interpretation.

2. *Objections to the Usual Viewpoint.*

Differentials arise quite early in the usual course of teaching the Calculus in schools: the meaning of a derivative is first explained, and then the differential dy is defined by the equation

$$dy = y' \delta x, \qquad \text{.......(1)}$$

where y' is the derivative of y with respect to x. Usually the textbook has an illustration to show the distinction between dy and δy, spends a page or two in labouring this distinction, and follows this up with a section on "Errors and Small Changes". It may also mention that the notation of differentials is due to Leibniz and that his symbol for the derivative, dy/dx, means dy divided by dx; though many modern textbooks prefer to introduce the symbol dy/dx first, and to derive the notation of differentials from it. Some students are able to accept this definition at its face value. Others are left with the impression that differentials are not real things at all; that dx and dy are *really* very minute changes in the variables, and always subject to a small degree of error (reminding him of the farmer who tried to train his horse to live on air by gradually reducing its daily feed, and almost succeeded, but unfortunately the animal died before the experiment was complete); and that the fuss about the difference between dy and δy is largely a subterfuge, an artificial device by which the writer can claim that his approximate equations are exact.

The subject is then dropped for a time, unless indeed differentials of higher orders are used to introduce the higher order derivatives, as in one widely-used school textbook. Here the student meets a difficulty: if he regards dx and dy as being exceedingly small, his brain reels at the idea of d^2y or d^3y; and if he has assimilated the figure of a curve and tangent to help him to under-

stand dy, he is offered no such aid to understanding d^2y, and his imagination boggles at the idea of a figure for d^3y. Fortunately he finds the idea of higher order *derivatives* quite easy: he is familiar with velocity and acceleration, and he can easily write down the second and third derivatives of the simple functions to which he is used: so he dismisses the talk of higher order differentials as mere fuss, and passes on to the next chapter.

Later he learns about integration as the limit of a sum. The fact that $\int y\,dx$ is the limit of $\Sigma y\,\delta x$ tends to strengthen the view that dx (which, the writer had originally maintained, could be any size he pleased if x were the independent variable) is *really* "indefinitely small". Examples from Physics confirm him in this belief: whenever an increment becomes very small it turns into a differential.

The difficulties so far have arisen because of the limitations of the student and his failure to get a clear idea of what a differential is, but when we come to partial differentiation the trouble is more serious. Differentials have been defined only in relation to a previously assigned independent variable, and although it is possible to extend this definition to a function of two variables and to give a geometrical interpretation, sooner or later the student is sure to meet a statement such as

$$\text{"}f(x, y) = 0, \quad \therefore \ \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0\text{,"} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and if his critical faculty has not been too far lulled to sleep by working with things he does not understand, he may reasonably be puzzled about the meaning of dx and dy in such a case. True, a meaning can be given on the lines that one of x and y is now dependent, and that (2) defines the differential of the dependent variable; but this presupposes the knowledge that

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y},$$

whereas (2) is often used to establish this fact. The student may reasonably feel dissatisfied with this and cling to the idea that dx and dy are "indefinitely small".

Here we reach the crux of the matter. If the definition of differentials distinguishes between dependent and independent variables, then the meaning of a given differential will change during the course of a problem in which the variable is regarded as dependent at one stage and independent at another. How then is a student to be convinced of the validity of the results? We can only tell him that an equation in differentials is formally true whatever variables be dependent or independent, and that therefore, although at any given stage of a proof he cannot attach any meaning to the differentials, nevertheless the final result will be formally true and he is at liberty to interpret it in the usual manner. This is deep water for a schoolboy, even if it be logically sound, and what will happen if higher order differentials are tackled seriously? Here we can no longer say that even the formal truth of an equation takes no regard of dependence and independence. No wonder higher order differentials are regarded as unsuitable for a school course. The student, failing to get a clear mental picture of d^2y, cannot fully understand the rules which he is obliged to remember, and will not avoid the pitfalls in which the subject abounds. How many candidates for University Entrance Scholarships (or, I venture to ask, how many school teachers) could say just why the first working of the following problem is wrong, and why the second is right?

Problem. If $y = \sin\theta$ and $x = \cos\theta$, express $\frac{d^2}{dx^2}(y^5)$ in terms of θ.

First "Solution" (wrong). As $y^5 = \sin^5\theta$, $d(y^5) = 5\sin^4\theta\cos\theta\,d\theta$.

$$\therefore\ d^2(y^5) = 20\sin^3\theta\cos^2\theta\,d\theta^2 - 5\sin^5\theta\,d\theta^2.$$

As $x = \cos\theta$, $dx = -\sin\theta\,d\theta$,

$$\therefore\ \frac{d^2}{dx^2}(y^5) = 20\sin\theta\cos^2\theta - 5\sin^3\theta = 5(4\sin\theta - 5\sin^3\theta).$$

Second Solution (right).

$$d^2(y^5) = (20\sin^3\theta\cos^2\theta - 5\sin^5\theta)\,d\theta^2 + 5\sin^4\theta\cos\theta\,d^2\theta.$$

But $dx = -\sin\theta\,d\theta$, and $d^2x = -\cos\theta\,d\theta^2 - \sin\theta\,d^2\theta$,

$$\therefore\ \sin^2\theta\,d\theta^2 = dx^2, \text{ and } \sin^3\theta\,d^2\theta = -\cos\theta\,dx^2 - \sin^2\theta\,d^2x,$$

$$\therefore\ d^2(y^5) = (20\sin\theta\cos^2\theta - 5\sin^3\theta - 5\sin\theta\cos^2\theta)\,dx^2;$$

$$\therefore\ \frac{d^2}{dx^2}(y^5) = 15\sin\theta\cos^2\theta - 5\sin^3\theta = 15\sin\theta - 20\sin^3\theta = 5\sin 3\theta.$$

3. *Suggested New Viewpoint.*

Consider the following two statements.

(*a*) The area of a rectangle of length l inches and breadth b inches is lb square inches.

(*b*) If A, B, C are three points in a straight line and $AC : CB = \lambda : \mu$, and if O is any point, then $\lambda\overline{OB} + \mu\overline{OA} = (\lambda + \mu)\overline{OC}$, where $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ are the vectors from O to A, B, C.

The first statement is still true if feet, miles, metres, or any other unit of length is read for inches throughout, and so for convenience the units are usually omitted, making the statement read: "The area of a rectangle of length l and breadth b is lb." In the second statement the position of O is arbitrary, and so it is usually not mentioned. The statement then reads: "If a, b, c are the position vectors of three collinear points A, B, C, and if $AC : CB = \lambda : \mu$, then $\lambda\mathrm{b} + \mu\mathrm{a} = (\lambda + \mu)\mathrm{c}$." These are two of many instances in which an arbitrary element is used to establish a general result, and in which the result is finally written in a form independent of the particular element chosen.

Now consider the following statement. "If x and y are two differentiable functions of a variable t, then

$$\frac{dy}{dt} = y'\frac{dx}{dt}, \qquad \text{.....(3)}$$

where y' is the derivative of y with respect to x." As the truth of the statement does not depend on the particular parameter t, the economic traditions of mathematics suggest that we should drop all mention of t and say: "If x and y are two related variables, and if y' is the derivative of y with respect to x, then

$$dy = y'\,dx." \qquad \text{.....(1)}$$

Here we have an entirely new definition of differentials, namely that "*Differentials are derivatives with respect to an arbitrary parameter, the same parameter being understood throughout all equations which are to be used simultaneously.*" In the earlier stages of teaching the subject it may be convenient to regard t as representing time, and the student will then be able to think of

differential as representing simply a rate of change. The definition is thus roughly equivalent to attaching Newton's idea of a fluxion to Leibniz' notation for a differential.

Later, we may approach the subject by a different route. Many teachers, when introducing the subject of limits and gradients, make it a practice to draw an enlarged picture of the graph which is being considered, showing δx and δy on a scale which makes them more readily visible to the class. Now if, as δx and δy tend to zero, we progressively magnify the picture so that they remain visible, this is equivalent to considering a third quantity δs which measures the length on the original figure represented by a unit length on the enlarged figure. When, in the limit, $\delta x \to 0$ in the original figure, its counterpart in the enlarged figure is $\lim_{\delta t \to 0} \frac{\delta x}{\delta t}$, or $\frac{dx}{dt}$. This explains why, in any physical problem, it is possible to pass from small increments to differentials.

It is better to leave higher order differentials until much later still, and to regard d^2y/dx^2 not as a quotient of differentials, but as a short way of writing $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. The derivative $\frac{dy}{dx}$ has been met primarily as a limit of $\frac{\delta y}{\delta x}$ and only secondarily as a quotient of differentials; and $\frac{d}{dx}$ has already arisen as a convenient notation when the function to be differentiated is too clumsy to go on the top line.

It will be noticed that differentials do not now occur until the "function-of-a-function" formula is reached. This is usually taught about the same time as other rules, such as that for differentiating a product, and our new definition lends itself readily to these formulae. Thus in the "product formula",

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the particular variable x is irrelevant, and we immediately abbreviate it to

$$d(uv) = v\,du + u\,dv, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

so that from our present viewpoint formulae (4) and (5) have exactly the same meaning.

At this stage it is convenient to point out to the student that any functional equation may be differentiated with respect to an arbitrary t, and to supply a number of examples for practice. This naturally brings in the usual problems about conical piles of sand, and water being poured into curiously shaped glasses, and leads easily to the subject of small errors. Problems on stationary values can be dealt with at the same time, and here the simplicity of our new definition of differentials is of great value, as the following example will show.

Example. Find r, given that $\pi r^2 h = 5$, and that $\pi r^2 + 2\pi rh$ is stationary.

We let $s = \pi r^2 + 2\pi rh$, and imagine that r and h both vary with respect to t (perhaps time) in such a way that $\pi r^2 h = 5$. Since r and h are both functions of t we can differentiate the equation

$$\pi r^2 h = 5, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and get $$2\pi rh\,dr + \pi r^2\,dh = 0, \text{ i.e. } 2h\,dr + r\,dh = 0. \quad \ldots\ldots\ldots\ldots (7)$$

Next we remember that by virtue of the equation

$$s = \pi r^2 + 2\pi rh, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

s is a function of t, and when it reaches its stationary value $ds = 0$. Hence for this value of t, by differentiating (8), $2\pi(r+h)\,dr + 2\pi r\,dh = 0$,

i.e. $$(r+h)\,dr + r\,dh = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Now dr and dh are not both zero, because a stationary value of s caused only by r and h being stationary is not what we are looking for, and so, eliminating them between (7) and (9), we get $2hr - r(r+h) = 0$, so that either $r = 0$ or $r = h$. The first of these is irrelevant, and so, substituting the second in (6), we get $r = \sqrt[3]{5/\pi}$.

We come next to the integral notation. In the expression $\int y\,dx$, shall the "dx" be regarded as a differential or not? Probably it is better to say at first that it is not a differential but merely a part of the operator $\int \ldots dx$, arising from $\Sigma \ldots \delta x$ in just the same way as dy/dx arose from $\delta y/\delta x$. After showing that

$$\int y\,dx = \int y \frac{dx}{dt}\,dt, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

it is possible to modify this attitude and say that "$\int$" and "d" are inverse operators, the first one meaning integration with respect to an arbitrary parameter, and the second, as before, differentiation with respect to the same parameter. This may be a little far-fetched, but it has at least the virtue of stressing the equivalence of the two integrals in (10), and it leads at once from (5) to the formula for integration by parts,

$$uv = \int v\,du + \int u\,dv. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Let us next consider partial differentiation. Here it might be supposed that a difficulty would arise: if u is a function of several independent variables x, y, ..., can all these variables be properly regarded as functions of *one* parameter t? It is clear that they can, because in any particular problem we are concerned with only a finite number of variables and with only a finite number of their derivatives; and functions of t can be found which will give to all of these any preassigned set of values. We start, then, in the usual way, by defining partial derivatives, and proceed to the fundamental formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

This immediately abbreviates to

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

and once again it must be stressed that these two equations have exactly the same meaning. In particular, if $f(x, y) = 0$, then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and the student should have no more difficulty in understanding the meaning of dx and dy in this case than in any other. Further, as our definition of differentials takes no account of which variables are independent and which dependent, no new complications are introduced by problems on change of independent variable, for the theorem on which they depend (namely that if

V is a function of $x, y, \ldots$ which are independent, and if $dV = A\,dx + B\,dy + \ldots$, then $A = \partial V/\partial x$, etc.) follows readily from this definition of differentials. If V is a function of u and v, which themselves are functions of x and y, then

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad \ldots\ldots(13)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy, \quad \ldots\ldots(14)$$

$$dV = \frac{\partial V}{\partial u}du + \frac{\partial V}{\partial v}dv$$

$$= \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}\right)dy. \quad \ldots\ldots(15)$$

Hence
$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}. \quad \ldots\ldots(16)$$

and a similar expression holds for $\partial V/\partial y$. This proof would not have been valid if we had followed the orthodox course of *defining* du by (13), dx and dy being arbitrary increments, for du would have had a quite different meaning in the first part of (15), where it would have been an actual increment in u. Hence the substitution would have been unjustified.

It remains for us to consider differentials of higher orders. Here a difficulty arises, not so much from the matter itself as from the present textbook conventions: the expression for d^2u in terms of dx differs according as x is or is not independent. This, as E. R. Hedrick remarks in his translation of *Cours d'Analyse* by Goursat, completely offsets the advantage enjoyed by first order differentials. It seems desirable, therefore, to reject this ambiguous definition and to regard d^2u as the second derivative of u with respect to an arbitrary parameter. In that case

$$d^2u = u''dx^2 + u'd^2x, \quad \ldots\ldots(17)$$

where dashes denote differentiation with respect to x, and it may immediately be objected that d^2u/dx^2 is now not the quotient of d^2u by dx^2. This drawback is inherent in Leibniz' notation for a derivative: it is not possible to regard d^2u/dx^2 as a quotient and at the same time so to define d^2u and dx^2 that they obey the ordinary laws of algebra, for this would imply that $\dfrac{d^2u}{dx^2}\left(\dfrac{dx}{dy}\right)^2$ was the same as $\dfrac{d^2u}{dy^2}$. Having admitted the difficulty, let us now observe in the first place that (17) is true whether or not x is independent, and, in the second, that the result of *formally* dividing through by dx^2 (which is equivalent to replacing t by x) is true, because $d^2x/dx^2 = 0$; and it is clear that such formal division of any equation in differentials must lead to a true equation in derivatives. It should perhaps here be pointed out that any equation in differentials must be homogeneous in their degree and order combined, for otherwise the equation will be altered when, say, $2t$ is substituted for t; this is true regardless of the number of variables, and determines what we shall call the *order* of the equation.

Dangers arise when the above process is reversed: from the true equation $d^2y/dx^2 = y''$ it does *not* follow that $d^2y = y''dx^2$, where the differentials are derivatives with respect to an arbitrary parameter. It must be conceded, though, that any convention is to be welcomed that makes it unnecessary to write down terms which will not be used in a particular instance. Several

conventions are possible, but we choose one which agrees with current practice, namely that "*If an equation of the nth order in differentials is to be used only for obtaining nth order derivatives we may omit all differentials of the independent variables of order higher than the first*", and we proceed to justify this convention.

To fix our ideas, we consider first a function u of two variables x and y, and confine ourselves to second order differentials. Then

$$d^2u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}dy^2 + \frac{\partial u}{\partial x}d^2x + \frac{\partial u}{\partial y}d^2y, \quad \ldots\ldots\ldots(18)$$

and this is true regardless of whether x and y are independent. If they are independent, and if by any other means we obtain

$$d^2u = A\,dx^2 + 2B\,dx\,dy + C\,dy^2 + E\,d^2x + F\,d^2y,$$

where A, B, ... , are functions of x and y, then

$$A = \frac{\partial^2 u}{\partial x^2}, \text{ etc.} \quad \ldots\ldots\ldots(19)$$

For by subtraction we obtain $\left(\frac{\partial^2 u}{\partial x^2} - A\right)dx^2 + \ldots = 0, \quad \ldots\ldots\ldots(20)$

where all these expressions are to be regarded as functions of t; and as it is possible to choose functions of t which shall, for a given value of t, give any preassigned values to x, y, dx, dy, d^2x, d^2y, it follows that the coefficients of the differentials in (20) must all be identically zero. It is evident that the principle may be extended to differentials of any order, and to functions of any number of variables. Now in the generalised form of (18), which is typical of the equations of the nth order referred to above, the nth order derivatives of u with respect to the independent variables are the coefficients of products of powers of the *first order* differentials of those variables, and this, by virtue of the theorem expressed in (19), is the basis of the above convention. This convention has been employed in the second solution of the problem at the end of Section 2, and explains the disappearance of d^2x in the fourth line. The error in the first solution is, of course, merely that $d^2(y^5)$ divided by dx^2 is not the same as $\frac{d^2}{dx^2}(y^5)$.

Perhaps enough has now been said to show that this definition of differentials is easily taught and easily understood, and that it gives an adequate interpretation wherever differentials may arise. There is no doubt that the differential notation has great power, and it is my hope that the material of this article may do something to make this power more readily accessible to students in the early stages of their work.

A. BARTON.

GLEANINGS FAR AND NEAR.

1466. I have compared the human beings in society to a great and increasing variety of colours tumultuously smashed up together, and giving at present a general and quite illusory effect of grey. . . . It is not a monotony, but an utterly disorderly and confusing variety that makes this grey, but Democracy, for practical purposes, does really assume such a monotony. Like ∞, the Democratic formula is a concrete-looking and negotiable symbol for a negation.—H. G. Wells, *Anticipations*, ch. 5. [Per Prof. E. H. Neville.]

SOME DIFFICULTIES IN TEACHING THE D METHOD IN LINEAR DIFFERENTIAL EQUATIONS.

By C. A. Coulson.

1. Certain steps in the solution of linear differential equations involving the operator $D \equiv \frac{d}{dx}$ nearly always cause the student some confusion. It may therefore sometimes be helpful to have an alternative way of proceeding. None of the results of this note is original, but some of the familiar difficulties are presented in a new light, which may prove useful, especially in a revision course.

2. The usual method of solution of the equation

$$\phi(D)\,y \equiv \{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n\}\,y = 0$$

is to try $y = Ae^{\lambda x}$, giving the auxiliary equation $\phi(\lambda) = 0$. If the roots of this equation are $\lambda_1, \lambda_2, \ldots \lambda_n$, the solution of the given equation is

$$y = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} + \ldots + A_n e^{\lambda_n x}.$$

The student finds three difficulties in this argument: first, the substitution $y = Ae^{\lambda x}$ seems to be "begging the answer"; second, the superposition of separate solutions; and third, the choice of, or restriction to, n arbitrary constants. These may all be overcome by a direct solution of the original equation as follows.

First consider the equation $(D - a)\,y = 0$. The student is already familiar with this in the form $dy/dx = ay$. We write it

$$\frac{dy}{y} = a\,dx,$$

and integrate both sides, giving $\log_e y = ax + \text{const.}$, *i.e.*

$$y = Ae^{ax}.$$

But we can get this result in quite a different way, which we may call the "operational method", since it only makes use of properties of the operator D. For, if $(D - a)\,y = 0$, then

$$\begin{aligned} y = \frac{1}{D-a}\cdot 0 &= \frac{1}{D}\left(1 - \frac{a}{D}\right)^{-1}\cdot 0 \\ &= \left(1 + \frac{a}{D} + \frac{a^2}{D^2} + \ldots\right)\cdot \frac{1}{D}\cdot 0 \\ &= \left(1 + \frac{a}{D} + \frac{a^2}{D^2} + \ldots\right) A \\ &= A\left\{1 + ax + \frac{a^2x^2}{2!} + \ldots\right\} \\ &= Ae^{ax}, \quad \text{where } A \text{ is an arbitrary constant.} \end{aligned}$$

If the student questions the validity of this operational expansion, we may argue alternatively

$$\begin{aligned} y = \frac{1}{D-a}\cdot 0 &= \frac{1}{D-a}(e^{ax}0) = e^{ax}\,\frac{1}{(D+a)-a}\,0 = e^{ax}\,\frac{1}{D}\,0 \\ &= Ae^{ax}. \end{aligned}$$

The operational method gives the same answer as the old familiar method. This should give us confidence in trying to solve harder equations in the same sort of way. Thus, if $(D-a)(D-b)y=0$, then

$$y=\frac{1}{(D-a)(D-b)}\cdot 0=\left\{\frac{C_1}{D-a}+\frac{C_2}{D-b}\right\}\cdot 0,$$

where C_1 and C_2 are certain constants depending on a and b. So by the previous result

$$y=Ae^{ax}+Be^{bx}, \quad A \text{ and } B \text{ being constants.}$$

In exactly the same way the solution of $(D-a)(D-b)\ \ldots\ (D-k)y=0$ is

$$y=Ae^{ax}+Be^{bx}+\ldots+Ke^{kx},$$

and thus all three difficulties referred to earlier are overcome. If two or more roots of the auxiliary equation are equal, the analysis requires slight modification : but this is easily provided.

3. Another difficulty often found is the case of complex roots $a\pm iw$ of the auxiliary equation. The clever student may enquire why it is necessary to introduce complex numbers and make use of the relation $e^{i\theta}=\cos\theta+i\sin\theta$ in order to end up with a purely " real " answer, $e^{ax}(A\sin wx+B\cos wx)$. The following analysis, which by-passes all mention of complex numbers, may be of interest. It is suggested by Note 1486 in No. 262 of the *Mathematical Gazette*, where it is shown that if

$$(D^2+w^2)y=0,$$

then

$$\begin{aligned}
y&=\frac{1}{D^2+w^2}\cdot 0\\
&=\left\{1-\frac{w^2}{D^2}+\frac{w^4}{D^4}-\ldots\right\}\frac{1}{D^2}\cdot 0\\
&=\left\{\ \ldots\ldots\ldots\ldots\ldots\ldots\ \right\}(Ax+B)\\
&=A\left(wx-\frac{w^3x^3}{3!}+\ldots\right)+B\left(1-\frac{w^2x^2}{2!}+\ldots\right)\\
&=A\sin wx+B\cos wx.
\end{aligned}$$

In the more general case, if

$$(D^2-2aD+b)y=0,$$

then

$$\begin{aligned}
y&=\frac{1}{D^2-2aD+b}\cdot 0=\frac{1}{(D-a)^2+w^2}\cdot 0, \text{ where } w^2=b-a^2,\\
&=\frac{1}{(D-a)^2+w^2}\cdot(e^{ax}0)\\
&=e^{ax}\frac{1}{D^2+w^2}\cdot 0\\
&=e^{ax}(A\sin wx+B\cos wx).
\end{aligned}$$

This shows that complex roots of the auxiliary equation which occur in pairs $a\pm iw$ give rise to corresponding terms $e^{ax}\cos wx$ and $e^{ax}\sin wx$.

4. Lastly, we come to equations in which there is a term $f(x)$ on the R.H.S. of the equation. The usual method is to separate the solution into

Complementary Function (C.F.) and Particular Integral (P.I.). But this distinction is nearly always confusing at first, and the clever student will be interested to see that all the technique used before will still apply ; the distinction between the C.F. and P.I. will appear quite naturally as the difference between the upper (variable) and lower (arbitrary constant) limits of an integral.

Thus, in the simplest case, if $(D-a)y=f(x)$, then

$$y=\frac{1}{D-a}\cdot f(x)=e^{ax}\frac{1}{D}e^{-ax}f(x)$$

$$=e^{ax}\int^{x}e^{-ax}f(x)\,dx.$$

The upper limit must be x : but the lower limit is quite arbitrary.

We can therefore write the solution in the alternative forms :

(i) $y=e^{ax}\int_{\alpha}^{x}e^{-ax}f(x)dx$, and

(ii) $y=e^{ax}\left\{\int^{x}e^{-ax}f(x)\,dx+C\right\}$,

where α and C are arbitrary constants. The C.F. is shown by the lower limit α in (i), and by the constant C in (ii).

This is easily extended to cover equations of higher order, and also cases in which the auxiliary equation has equal roots. We give the result for k factors, all different.

If $\phi(D)y\equiv(D-a)(D-b)\ldots(D-k)y=f(x)$, then

$$y=\frac{1}{\phi(D)}\cdot f(x)=\left\{\frac{C_1}{D-a}+\frac{C_2}{D-b}+\ldots+\frac{C_k}{D-k}\right\}f(x),$$

where $C_1=1/\phi'(a)$, $C_2=1/\phi'(b)$, ... etc. So the solution is

$$y=C_1e^{ax}\int_{x_1}^{x}e^{-ax}f(x)\,dx+\ldots+C_ke^{kx}\int_{x_k}^{x}e^{-kx}f(x)\,dx,$$

or, in equivalent form,

$$y=C_1e^{ax}\int^{x}e^{-ax}f(x)\,dx+\ldots+Ae^{ax}+Be^{bx}+\ldots+Ke^{kx}.$$

The distinction between C.F. and P.I. is here revealed in a new light.

5. In conclusion, it may be argued that there is a certain advantage in the sequence illustrated above. For it represents a complete operational calculus—the simplest commonly required—and a student trained to appreciate this scheme will be more ready to understand the more complicated operational methods (Heaviside, Laplace, etc.) that he will meet with in a University Honours Course. C. A. C.

1467. We climbed Snowdon up the Watkin track, and when we came to the smooth dome of the peak we found that it wore a cap of solid ice.... Had we slipped no earthly power could have saved us. My fingers felt at the smooth, hostile slope, and my boot-nails barely scratched the surface. I stayed where I was by so little friction that I verily believe heavier clothes would have sent me plunging down.—Thomas Firbank, *I Bought a Mountain* (Harrap, 1940), page 81. [Per Mr. F. W. Kellaway.]

SIMPLE NOMOGRAMS.

By J. H. Cadwell.

While a full discussion of nomograms requires the use of determinants and other ideas beyond the School Certificate stage, many practical needs can be met using two simple types. These can be understood and applied with only a knowledge of similar triangles ; and, besides being of interest in themselves, produce as a by-product a working knowledge of logarithmic and other scales.

Before considering the method of construction, use of one or two types should be made by the pupil. This will help him to appreciate their simplicity. Their practical value, too, becomes apparent by using types (*e.g.* the focal length of a lens) that tie up with his work in the laboratory. Construction on squared paper can be done quickly and with fair accuracy. One or two carefully done examples with subdivisions inserted should be attempted also.

1. *Addition.*

The basic addition nomogram is illustrated in Fig. 1, and the proof is as follows.

Since (*c*) scale is twice the (*b*) scale, and by using similar triangles, we get :

NC measured in (*c*) scale = MB measured in (*b*) scale = b,

and Scale reading of N = Scale reading of A = a.

Hence Scale reading of $C = a + b$,

and therefore $c = a + b$.

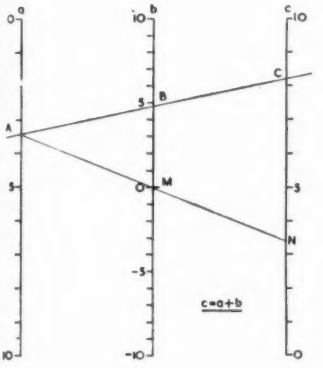


Fig. 1.

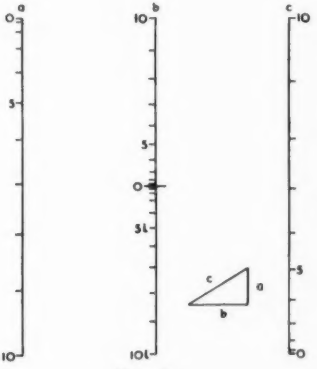


Fig. 2.

Using this arrangement of scales most practical cases can be covered, and Figs. 2 and 3 illustrate two applications. Note that in Fig. 3

$$\log V = \log \pi + 2 \log r + \log h \quad \text{or} \quad V = \pi r^2 h,$$

and we plot $2 \log r$ on the (r) scale. The term $\log \pi$ is allowed for in the choice of the unit value on the (h) scale.

Where the range of values considered makes a different relative scaling advisable, one can avoid the calculation of this scaling as follows. Plot (*a*) and (*b*) to suitable arbitrary scales and complete the (*c*) scale by selected problems.

2. *Multiplication.*

This is illustrated in Fig. 4, where for purposes of introduction the (b) scale is regarded as the projection of the extended (c) scale. In practice we avoid this extension by selected problems.

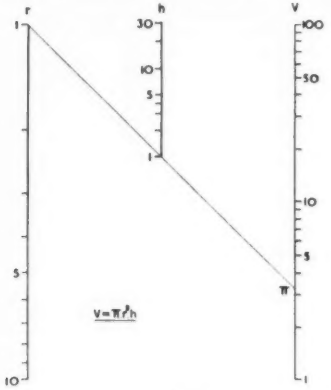


FIG. 3.

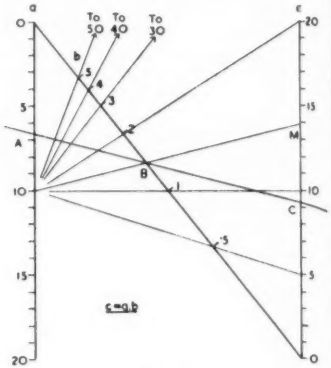


FIG. 4.

By the method of construction we have

$$b = \text{Scale reading of } B = \tfrac{1}{10} \text{ Scale reading of } M.$$

By similar triangles,

$$\frac{\text{Scale reading of } C}{\text{Scale reading of } A} = \frac{\text{Scale reading of } M}{10},$$

or $c/a = b$, and so $c = ab$.

As we now graduate the (b) scale by using the other two, these can be scaled arbitrarily to suit the ranges of values involved. Note that in the case of Fig. 5 ($c = a^b$) the graduation of the (b) scale is best carried out using the underlying linear scales instead of values of a and c.

3. *Further Work.*

To cope with formulae containing four variables, a set-square nomogram may be of use. This is illustrated in Fig. 6. The points A and B are aligned, and the set-square moved till its edge perpendicular to AB goes through C. This edge then cuts the Δ scale at D in the value sought.

The underlying form here is $a \,.\, d = b \,.\, c$; this is used in Fig. 6 as

$$2\Delta \operatorname{cosec} A = b \,.\, c.$$

Scaling of (b), (c) and (Δ) is arbitrary, that of the fourth scale then follows readily, either by calculation or using selected problems.

The principle of repeated alignments is illustrated by

$$(1/r_1) + (1/r_2) + (1/r_3) = 1/R.$$

This is split into

$$(1/r_1) + (1/r_2) = x, \quad x + (1/r_3) = 1/R,$$

and two nomograms with a common (x) scale are made. In this case we can arrange for the (R) and (r_1) scales to coincide also, thus getting a very compact

result (Fig. 7). By superimposing Figs. 3 and 5 we could with one alignment of a and b get

$$c = \pi a^2 b, \qquad c' = a^b.$$

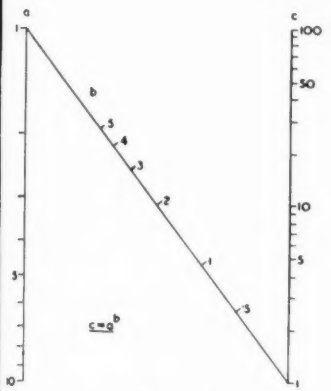


FIG. 5.

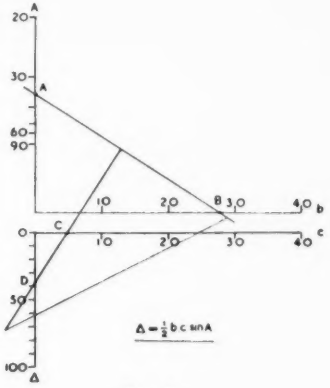


FIG. 6.

A less trivial simultaneous nomogram (Fig. 8) is for the four quantities a, b, c and A, associated with a right-angled triangle. These follow from

$$c^2 = a^2 + b^2, \qquad a^2 = c^2 \sin^2 A.$$

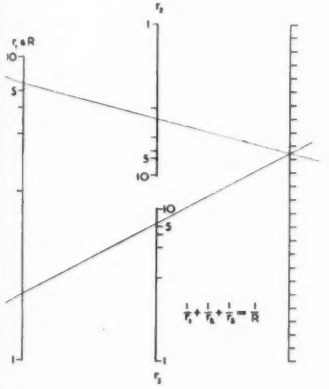


FIG. 7.

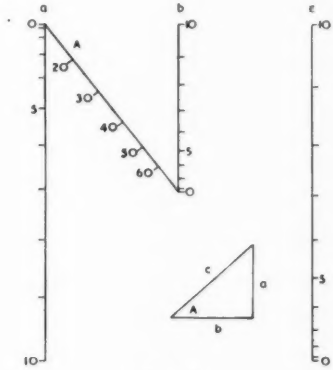


FIG. 8.

There are many other directions in which the subject can be developed, but they require methods beyond those presupposed in this article. However, practical work in the use and construction of curved scale nomograms is within the capacity of the fourth former, and provides a variety of training.

The figures in this article make no pretensions to accuracy, and only main divisions have been inserted.

J. H. C.

AMPLITUDE OF TRIODE OSCILLATIONS.

By F. B. Pidduck.

By a simplifying assumption we express amplitude, reaction, frequency, and change of plate current in triode oscillations in terms of the fraction of the time during which plate current is flowing. The mathematical methods are simple, and were used by E. V. Appleton and B. van der Pol, *Phil. Mag.*, 43, 177, 1922.

A triode is an exhausted glass vessel containing three electrodes : a filament or heater, a grid, and a plate. Electrons emitted by the heater are caught on the grid and plate in numbers depending on their potentials, and constitute currents from the leading-in wires towards those electrodes, called the grid current and plate current. If V_g is the potential of the grid relative to the heater, and V_p that of the plate, the plate current I_p is theoretically and experimentally a function $f(V_p+\mu V_g)$ of $V_p+\mu V_g$ over a large range, where μ is a constant called the amplification. Thus the curves of I_p for different plate potentials V_p are obtained from each other by translation parallel to the axis of V_g by an amount proportional to the change of V_p. The grid current is much smaller if the grid potential is not too high.

Consider a triode maintaining oscillations in a tuned plate circuit (Fig. 1).

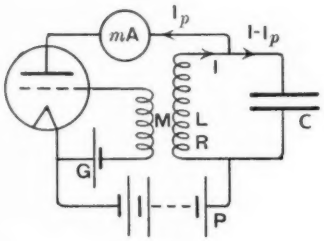


Fig. 1.

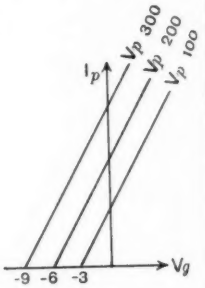


Fig. 2.

Let the grid circuit contain a battery of potential G (usually from -30 to 0 volts) and the plate circuit a battery of potential P (from 100 to 1000 volts). If M is the mutual inductance between the grid coil and the plate coil L, and if the plate coil has resistance R, the circuit equations are

$$I-I_p=CdV_p/dt, \quad V_p=P-LdI/dt-RI, \quad V_g=G-MdI/dt, \quad \ldots\ldots(1)$$

where grid current has been neglected. From what has been said

$$I_p=f(V+NdI/dt-RI),$$

where

$$V=P+\mu G, \quad N=-L-\mu M. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Eliminating I_p, V_p, and V_g, we have

$$LC\frac{d^2I}{dt^2}+RC\frac{dI}{dt}-f\left(V+N\frac{dI}{dt}-RI\right)+I=0.$$

The natural oscillations are in general damped out slowly, and the triode has only to make up that damping. Hence the two middle terms of the last

equation are small. If they are at first put zero, $I = I_0 \sin \omega t$, where $\omega^2 = 1/LC$. We call $\omega/2\pi$ the natural frequency of the LC circuit. Substituting this expression for I in the middle terms we have to a next approximation

$$d^2I/dt^2 + \omega^2 I = -\omega^2 \phi(\omega t), \quad \ldots\ldots(3)$$

where $$\phi(\omega t) = RC\omega I_0 \cos \omega t - f(V + N\omega I_0 \cos \omega t - RI_0 \sin \omega t). \quad \ldots\ldots(4)$$

Speaking generally, the oscillations are damped out when the characteristic is horizontal ($I_p = 0$) and increased when it is oblique. We can obtain compact formulae by taking the characteristics to be bent straight lines, or

$$f(x) = 0 \text{ if } x < q, \; f(x) = p(x - q) \text{ if } x > q. \quad \ldots\ldots(5)$$

These characteristics are shown with a few typical numbers in Fig. 2, and form a rough approximation to the curved characteristics given in makers' catalogues. Any theory of "mutual conductance" or "plate resistance" assumes the characteristics to be straight over at least a small region, but will not determine the amplitude if that region is finite. The solution of (3) which adds nothing to I or $dI/d\omega t$ when $t = 0$ is

$$I = I_0 \sin \omega t - \int_0^{\omega t} \phi(\omega\tau) \sin(\omega t - \omega\tau)\, d\omega\tau. \quad \ldots\ldots(6)$$

If $I = I_0 \sin \omega t$, the oscillation passes from an oblique segment to a horizontal segment when $V + N\omega I_0 \cos \omega t - RI_0 \sin \omega t - q = 0$. Write

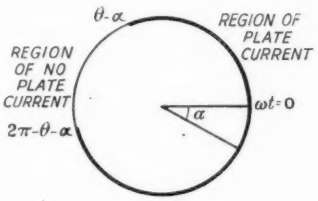


FIG. 3.

$$N\omega = \rho \cos \alpha \text{ and } R = \rho \sin \alpha.$$

Then if θ is the smallest positive root of the equation $\rho I_0 \cos \theta + V - q = 0$, the oscillations are on an oblique segment from $\omega t = 0$ to $\theta - \alpha$, on a horizontal segment from $\theta - \alpha$ to $2\pi - \theta - \alpha$, and on an oblique segment round again to $2\pi + \theta - \alpha$. The plate current flows for a fraction

$$f = \theta/\pi \quad \ldots\ldots(7)$$

of the whole period. If the oscillation repeats itself, it will be in a time such that ωt differs only slightly from 2π, say $2\pi(1 - \eta)$. In this case I and $dI/d\omega t$ have the same values 0 and I_0 when $\omega t = 0$ and $2\pi(1 - \eta)$. From (6),

$$I = A - 2\pi(I_0 + B)\eta \text{ and } dI/d\omega t = I_0 + B + 2\pi A\eta$$

when $\omega t = 2\pi(1 - \eta)$, where

$$A = \int_0^{2\pi} \phi(\omega\tau) \sin \omega\tau\, d\omega\tau, \quad B = -\int_0^{2\pi} \phi(\omega\tau) \cos \omega\tau\, d\omega\tau. \quad \ldots\ldots(8)$$

The oscillation repeats itself if $A - 2\pi(I_0 + B)\eta = B + 2\pi A\eta = 0$. Here η would

be finite if B was really of the first order, so that we must have $B=0$ to this order. From (4), (5), and (8),

$$RC\omega I_0\pi = p\int_{-\theta-\alpha}^{\theta-\alpha}[V-q+N\omega I_0\cos\omega\tau - RI_0\sin\omega\tau]\cos\omega\tau\, d\omega\tau,$$

which reduces to

$$2RC\pi/pN = 2\theta - \sin 2\theta. \quad \ldots\ldots(9)$$

$N\omega$ is usually large compared with R, and ρ can be put equal to $N\omega$. Then

$$N\omega I_0\cos\theta = -(V-q). \quad \ldots\ldots(10)$$

Equations (7), (9), and (10) give N and I_0 in terms of f (see below). The frequency is given by $A - 2\pi I_0\eta = 0$. Evaluating A, we find

$$4\pi\eta/pR = 2\theta - \sin 2\theta. \quad \ldots\ldots(11)$$

The frequency is higher than the natural frequency of the LC circuit in the ratio of $1+\eta$ to 1.

A convenient test of oscillation, independent of any thermal ammeter in the LC circuit, is a D.C. milliammeter in the plate circuit (Fig. 1), which changes its deflexion when oscillations begin. Let us calculate the amount of this change. The plate current at time t is approximately

$$f(V + N\omega I_0\cos\omega t - RI_0\sin\omega t),$$

and its mean value during a period is, to the first order,

$$\frac{1}{2\pi}\int_0^{2\pi} f(V+N\omega I_0\cos\omega\tau - RI_0\sin\omega\tau)\,d\omega\tau$$

$$= \frac{p}{2\pi}\int_{-\theta-\alpha}^{\theta-\alpha}(V-q+N\omega I_0\cos\omega\tau - RI_0\sin\omega\tau)\,d\omega\tau$$

$$= p\rho I_0(-\theta\cos\theta+\sin\theta)/\pi.$$

The steady potentials P and G are usually such that $V>q$, and the plate current before oscillations begin is then $p(V-q) = -p\rho I_0\cos\theta$. Thus the change of plate current when oscillations begin is

$$\delta I_p = pN\omega I_0(\pi\cos\theta - \theta\cos\theta+\sin\theta)/\pi, \quad \ldots\ldots(12)$$

where ρ has been replaced by $N\omega$ as before.

From equations (7) to (12), pN/RC, $\omega RCI_0/p(V-q)$, η/pR, and $\delta I_p/p(V-q)$ are functions of f, which we now tabulate.

f	$\frac{pN}{RC}$	$\frac{\omega RCI_0}{p(V-q)}$	$\frac{\eta}{pR}$	$\frac{\delta I_p}{p(V-q)}$	f	$\frac{pN}{RC}$	$\frac{\omega RCI_0}{p(V-q)}$	$\frac{\eta}{pR}$	$\frac{\delta I_p}{p(V-q)}$
1·00	1	1	0·500	0	0·78	1·068	1·213	0·468	0·286
0·98	1	1·002	0·500	0·022	0·76	1·088	1·260	0·459	0·323
0·96	1	1·007	0·500	0·044	0·74	1·112	1·314	0·449	0·365
0·94	1·001	1·016	0·499	0·067	0·72	1·141	1·375	0·438	0·413
0·92	1·003	1·029	0·498	0·090	0·70	1·175	1·449	0·426	0·468
0·90	1·007	1·045	0·497	0·114	0·68	1·213	1·538	0·412	0·534
0·88	1·011	1·064	0·495	0·138	0·66	1·259	1·651	0·397	0·614
0·86	1·018	1·086	0·491	0·164	0·64	1·311	1·789	0·381	0·712
0·84	1·026	1·110	0·487	0·191	0·62	1·372	1·980	0·364	0·842
0·82	1·037	1·142	0·482	0·220	0·60	1·442	2·244	0·347	1·019
0·80	1·051	1·176	0·476	0·251	0·58	1·523	2·641	0·328	1·282

The oscillations in this simple theory start suddenly at a certain value of N. This is more or less what is found by experiment. If V alone is varied, I_0 is proportional to $V - q$, which is nearly proportional to V. The maximum potential across the condenser is $L\omega I_0$, which is also nearly proportional to V. If C is varied, the value of N required to start oscillations is proportional to C, and I_0 is proportional to $1/\omega C$ or $C^{\frac{1}{2}}$.

For a numerical example take $p = 1/12000$, $q = 20$, $\mu = 30$, $P = 200V$, $G = -3V$, $R = 2\Omega$, $L = 50\mu H$, and $C = 1m\mu F$, which gives $\omega = 4{\cdot}47 \times 10^6$ or a wave-length of $420m$. To start oscillations requires N to be $24\mu H$, or $M = -2{\cdot}47\mu H$. This makes $N\omega/R$, which was assumed to be large, equal to 53. The oscillatory current I_0 is then $0{\cdot}84A$, and η is 1/12000. If M is raised to $-2{\cdot}6\mu H$, $I_0 = 1{\cdot}16A$, $\eta = 1/14000$, and $\delta I_p = 3mA$. At this stage plate current is flowing for 72 per cent. of the time. The maximum potential across the condenser is $270V$. The theory is of course only qualitative.

F. B. PIDDUCK.

1468. It will be proper to add to this brief sketch something about her feeling for this country. Because she was not English by birth, she was able to examine us with something of the detachment of the foreigner, though with the steady fellow-feeling which belongs to the tie of blood and Dominion tradition. She was able to see both our virtues and our follies in dealing with the rest of the Commonwealth, and would have done valuable work in interpreting Britain and Australia to each other. She had done much already, both by her books and in visits to her native continent. Nobody, I suppose, who has not lived on both sides of the Equator can fully realise the curious dislocations of outlook that occur when they contemplate one another. "You can't imagine," she would say, "how *odd*, for instance, all European poetry becomes when you live in a country where the sun rises in the West." Indeed, a transplantation from the Antipodes must work to clear the mind of preconceptions, and the Australian may be excused a natural impatience with the kind of English statesmanship which assumes, without consideration, that the sun can rise nowhere but in the East.—D. L. Sayers, "Helen Simpson", *The Fortnightly*, January, 1941. [Per Prof. H. S. Carslaw.]

1469. ". . . I cannot bestow a better blessing on mankind than sincerely to wish that he would . . . undertake a work which he likewise proposes to execute, namely, by corollaries and illustrations to render the study of Mathematics pleasurable and interesting, instead of being what it now is, almost universally to young beginners, dry, tedious and disgusting."—From *A Pedestrian's Diary through part of the Highlands of Scotland in* 1801, by John Bristed, Vol. I, page lxxiv (Wallis, 1803). The quotation is from the Preface commenting on *Anthropaideia, or a Tractate on General Education*, by Andrew Cowan, M.D. London, 1803. (Wallis.) [Per Dr. H. Lowery.]

1470. I mean, he replied, as you might say of the very large and very small — that nothing is more uncommon than a very large or very small man ; and this applies generally to all extremes, whether of great and small, or slow and swift, or fair and foul, or black and white, and whether the instances you select be men or dogs or anything else, few are the extremes, but those betwixt and between are numberless.—Socrates, as recorded in Plato's *Phaedo*, 39, tr. Jowett.

Is this the first record of the idea of the bell-shaped or "cocked hat" frequency distribution, now so common in statistics? [Per Dr. G. J. Lidstone.]

HARMONIC POLYGONS.

By T. Sharp.

Browsing for amusement in a book that I never saw at school, Casey's *Sequel to Euclid*, I was introduced to something new to me. It was my first encounter with harmonic polygons, and it was interesting to trace how the painstaking geometers built up a series of propositions applicable to such figures. What annoyed me, however, was the attempt to derive some of their properties by "inversion" of a regular polygon. Again it seemed illogical to build up a proof of general properties by proving a succession of mensurations. This feeling was intensified when it came to deal with the "Brocard" properties and I started to build up the theory from a new angle.

First of all, an all-embracing definition was needed which could be reduced to a few words, and I finally decided on :

A harmonic polygon is the figure obtained by projecting a regular polygon.

It is comparatively simple to show that this definition embraces every other ; and it is nearly as simple to show that a polygon defined as harmonic by any other general property falls within the above definition.

To arrive at the properties of a harmonic polygon we simply recast the properties of a regular polygon as obtained by a simple projection : *e.g.*

1. The vertices lie on a conic.
2. The sides touch a conic.
3. The two conics have double contact.
4. The two conics have a common polar (p) and a common pole (K).
5. If the polygon have $2n$ sides, then the diameters pass through K and the opposite sides meet on p.
6. If the polygon have $2m+1$ sides, then the join of any vertex to the point of contact of the opposite side with the inconic passes through K, and the intersection of any side with the tangent (to the circumconic) at the opposite vertex is on p.
7. Porism applies. The polygon can, as it were, be rotated, and all such polygons are associated with K and p.
8. Any side is divided harmonically at the point of contact and the intersection with p.
9. A pair of sides through a vertex, with the tangent to the circumconic at that vertex, and the join of the vertex to K, form a harmonic pencil.
10. Either conic can be a circle ; but if both are circles they must be concentric.
11. If either conic be a circle, mensuration properties can be deduced.
12. If we call the lines in (5) and in (6) through K diameters of the polygon, then the diameters and p comprise the locus of the intersection of the $n(n-1)/2$ lines joining the n vertices.
13. The n diameters have n poles on p, and these (with K) form $n+1$ points through which pass the lines joining any two intersections of the n tangents, or sides.
14. The successive vertices, on quasi-rotation of the polygon, form projective ranges on the circumconic.
15. A similar property for the sides.
16. The diameter property shows that we can divide up the vertices into a number of projective ranges (in fact, either $2n+2$ or $2n+1$ different sets), and similarly for the sides.
17. If the parameter of a point on the circumconic be l and on the inconic be m, then l and l_r are connected by a linear relation, wherever l may be

understanding that l_r is the vertex obtained by drawing r tangents in succession to the inconic in a definite direction, say clockwise).

18. A similar property for the parameters of the sides regarded as tangents to the inconic.

19. If we start from a point A on the circumconic and draw a pair of tangents to the inconic cutting the circumconic in a pair of vertices B and B', then the parameters of A and B are connected by a quadratic relation, which *must* break down into two linear equations.

20. If, continuing the process, we obtain vertices C and C', then we get *another* quadratic relation, which also breaks up into linear factors, ..., and so on for the whole series.

21. Similar relations for the parameters of the tangents to the inconic.

22. The linear transformation for successive vertices (or sides) after n operations brings us back to starting point.

23. If T_r is the transformation corresponding to r tangents or operations in succession, (whether or not r is prime to n), then n operations, each equivalent to T_r, brings us back to the starting point. These operations give rise to subsidiary harmonic polygons.

24. It will be convenient to denote by T the (linear) transformation from any vertex to the opposite vertex, which occurs when we have $2n$ sides.

25. Similarly, if we have $2n+1$ sides, there will be a linear relation between the parameter of any vertex and the parameter of the opposite side. Call this transformation T'.

The ideas underlying the use of T and T' will be particularly useful in investigating the property of porism.

26. The points of contact of the sides form a harmonic polygon, and so do the tangents to the circumconic at the vertices ... , and so on *ad infinitum*.

Three general observations here arise. *First*, the reverse process of working back from any "harmonic" property of a given polygon so as to show that polygon falls within the general definition is comparatively simple, and is a matter of ordinary projective geometry. . *Second*, if we pursue the path of projective geometry, then the next logical step to take is to generalise the Brocard Circle ; and *third*, there is a property already touched upon which does not properly belong to our general investigation, namely, *porism*. Perhaps we should turn aside to investigate.

Porism.

Suppose that we have a conic inscribed in one conic and circumscribed to another. We can think of this polygon as formed by starting from a particular vertex and drawing a pair of tangents to the inconic, thereby arriving at points B and B'. Now we do *not* know that the quadratic relation between the parameters breaks up into linear factors. We proceed. Obtain C and C'. If the number of sides be $2n$, then the nth operation gives rise to a quadratic with equal roots, and the $2n$th operation gives rise to a quadratic with equal roots each equal to the original parameter (*i.e.* a linear relationship).

In either event, every quadratic relationship is invariable in *form*, so that precisely the same result will follow if we start with any point whatsoever on the circumconic.

If the sides be $2n+1$, then take $2n+1$ operations.

In other words, *porism always obtains*.

Next, if porism obtains, must the polygon be harmonic? Now we know that in the case of the triangle, four-point or five-point, we can always draw an inconic and a circumconic, whether or not the polygon is harmonic, but in these cases we also know that the diameters or quasi-diameters co-intersect at a point corresponding to K.

Again, if n equals 6, the application of Pascal's theorem and Brianchon's theorem establish the existence of the point K and *we have enough data* to establish two projective ranges on the circumconic, which, in turn, establish the harmonic properties.

If n is 3, 4 or 5, we have *not* enough data to establish two projective ranges on the circumconic and it cannot be proven that the figures are harmonic (although it is possible).

If n is greater than 5 *and* we can establish the existence of the point K, *then* we also have enough data to set up two projective ranges and all harmonic properties follow.

We apply therefore the ideas outlined in paragraphs 24 and 25. The idea we use depends on whether the number of sides is even or odd. We do not know that any of the transformations T_1, T_2, etc., will break down into linear factors, but *we do know* that if we carry on until we reach T (the $2n/2$) transformation we get one with equal and therefore linear roots. This establishes a one-to-one correspondence between any point and the so-called nth point and determines the existence of the point K.

In the case of $2n+1$ sides, the operation T' establishes a one-to-one correspondence between any point and the point of contact of the nth tangents (which are the same whether we go clockwise or counterclockwise). If we produce this quasi-diameter to meet the circumconic, we get a true one-to-one correspondence on the *same* conic and determine the existence of the point K. This process merely amounts to expanding the polygon of $2n+1$ sides into a polygon of $4n+2$ sides.

We reach therefore a rather remarkable result : *If n is greater than 5, and a polygon of N sides is such that the vertices lie on a conic and the sides touch another conic, then the polygon is harmonic* ... porism applies, but as a special case of :

If a polygon be such that the vertices lie on one conic and the sides touch another, then an infinite number of such polygons can be drawn.

The Brocard Conic.

For simplicity, take a harmonic polygon ABC ... such that conic ABC ... is an ellipse. So also will be the inconic. Choose any point O within the conic ABC ... and draw the polar (l) of O with respect to the circumconic ABC ... .

We have also the common pole or quasi-symmedian point K and its polar p. The circumconic cuts p in two imaginary points S, S', and will cut l in two other imaginary points I and J.

Let AB cut l in a', BC cut l in b', etc., and consider the two pencils $K(a'b'c'$ ...) and $O(abc$ ...), where Oa is the polar of a', Ob the polar of b', etc. Corresponding rays of these two pencils intersect on a conic through O and K, which will also pass through I and J, and also through S and S'.

This is a Brocard Conic.

The pencil $S_1(a'AaB)$ is harmonic and can be treated as transforming into pencil $S_2(b'BbC)$, etc., etc.

We can call S_1, S_2 ... the foci.

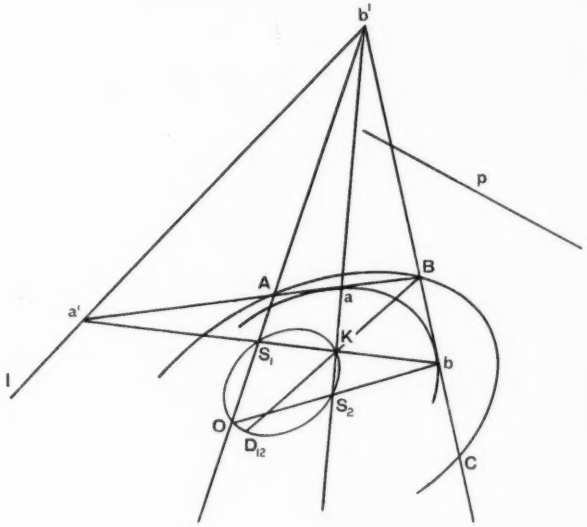
The pencil $a'K$, $a'AB$, and 1 are three rays of a pencil through a' which can be treated as transforming into $b'K$, $b'BC$ and 1.

All points and *lines* of any geometrical figure associated with AB can be thought of as built up from the two pencils of that figure and will transform into points and lines associated with BC, etc.

Any conic associated with AB will transform into a conic that is associated with BC, and in fact any curve of mth degree and nth order associated with

AB will transform into a curve of similar degree and order associated with BC.

In particular, corresponding lines through the foci S_1 and S_2 must intersect on a conic through those foci, and O and K and I and J ... , that is to say on the Brocard Conic.



Again, corresponding lines through a' and b' intersect on KB, and if KB cuts the Brocard conic in D_{12}, then this point *must* be a double point of the two configurations.

On investigation we find that I and J are also double points.

By projecting I and J into the circular points at infinity we get the case of the Brocard *circle*, in which event the transformations give rise to *similar* figures. T. SHARP.

1471. Sir James Grigg was in reminiscent mood yesterday. The occasion demanded it, for he was the guest of honour at a lunch of the Johnian Society which was attended by about 130 past and present members of St. John's College, Cambridge.

Talking of the college worthies of his day, Sir James, who was a mathematical scholar, said how he had read with R. R. Webb. That famous coach one year had seventeen pupils, and the lowest of them in the Tripos list was the eighteenth wrangler.

Fondly imagining himself to be omniscient, the future War Secretary attempted to answer a question put him by Webb. His coach thereupon tore off the corner of a sheet of paper, handed the scrap to him and said, "Mr. Grigg, write all you know about mathematics on that."—*The Daily Telegraph*. [College legend asserts that the paper was the back of a postage stamp.]

AN ASYMPTOTIC SERIES FOR COMPUTING π.

By A. P. Guinand.

1. *Introduction.*

The series

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \ldots\ldots(1)$$

is the simplest known series * which could be used to compute π, but it converges far too slowly to be of any practical use for this purpose. In this paper we prove an asymptotic formula for the remainder of the series (1) after a finite number of terms. The formula is

$$\frac{1}{4}\pi = \sum_{n=0}^{2N-1} \frac{(-1)^n}{2n+1} + \frac{1}{2}\sum_{r=0}^{k} \frac{E_{2r}}{(4N)^{2r+1}} + R_k, \quad \ldots\ldots(2)$$

where

$$|R_k| < \frac{(4k+5)(2k+1)!}{(2\pi N)^{2k+3}}. \quad \ldots\ldots(3)$$

The E_r are the Euler numbers † which occur in the expansion

$$\operatorname{sech} z = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

This formula readily gives π to considerable accuracy, and it can be proved for any specific value of k by very elementary methods.

2. *Elementary proof when* $k = 1$.

From (1),

$$\frac{1}{4}\pi = \sum_{n=0}^{2N-1} \frac{(-1)^n}{2n+1} + \sum_{n=2N}^{\infty} \frac{(-1)^n}{2n+1}. \quad \ldots\ldots(4)$$

Now

$$\begin{aligned}\sum_{n=2N}^{\infty} \frac{(-1)^n}{2n+1} &= \frac{1}{4N+1} - \frac{1}{4N+3} + \frac{1}{4N+5} - \ldots \\ &= \sum_{n=N}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n+3}\right) \\ &= \sum_{n=N}^{\infty} \frac{2}{(4n+1)(4n+3)}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{n=2N}^{\infty} \frac{(-1)^n}{2n+1} - \frac{1}{8N} + \frac{1}{128N^3} \\ &= \sum_{n=N}^{\infty} \frac{2}{(4n+1)(4n+3)} - \frac{1}{8}\sum_{n=N}^{\infty}\left\{\frac{1}{n} - \frac{1}{n+1}\right\} + \frac{1}{128}\sum_{n=N}^{\infty}\left\{\frac{1}{n^3} - \frac{1}{(n+1)^3}\right\} \\ &= \sum_{n=N}^{\infty} \frac{256n^3(n+1)^3 - 16n^2(n+1)^2(4n+1)(4n+3) + (4n+1)(4n+3)(3n^2+3n+1)}{128n^3(n+1)^3(4n+1)(4n+3)} \\ &= \sum_{n=N}^{\infty} \frac{25n^2+25n+3}{128n^3(n+1)^3(4n+1)(4n+3)}. \quad \ldots\ldots(5)\end{aligned}$$

* E. W. Hobson, *Squaring the Circle* (Cambridge, 1913), gives many series which can be used to compute π.

† K. Knopp, *Theory and Application of Infinite Series* (London, 1928), 239.

Now, if $n \geqslant 2$, then $25n^2 + 25n + 3 < 40n^2$, and the series (5) is less than

$$\frac{40}{128 \times 16} \sum_{n=N}^{\infty} n^{-6} < \frac{5}{256} \int_{N-1}^{\infty} t^{-6}\, dt = \frac{(N-1)^{-5}}{256}.$$

Further, all the terms of the series (5) are positive. Hence, by (4),

$$0 < \frac{1}{4}\pi - \left\{ \sum_{n=0}^{2N-1} \frac{(-1)^n}{2n+1} + \frac{1}{8N} - \frac{1}{128N^3} \right\} < \frac{(N-1)^{-5}}{256}$$

if $N \geqslant 2$. This result corresponds to (2) when $k = 1$, and the method can be extended to other values of k. However, the algebra becomes rather involved if we use this method to prove the formula for general k, and it is more convenient to use another method.

3. *Proof for general values of k.*

Let $\phi(x)$ be a periodic function, of period 4, defined by

$$\phi(x) = \begin{cases} x & (0 \leqslant x \leqslant 1), \\ 2 - x & (1 \leqslant x \leqslant 3), \\ x - 4 & (3 \leqslant x \leqslant 4), \end{cases}$$

and

$$\phi(x + 4n) = \phi(x).$$

The Fourier series for $\phi(x)$ is

$$\phi(x) = 2\left(\frac{2}{\pi}\right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sin \frac{1}{2}\pi(2n+1)x. \qquad \text{.....(6)}$$

If we put

$$\chi(n) = \begin{cases} 0 & (n \text{ even}), \\ 1 & (n = 4m+1), \\ -1 & (n = 4m-1), \end{cases}$$

where m and n are integers, then (6) may be written

$$\phi(x) = 2\left(\frac{2}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} \sin \frac{1}{2}\pi n x. \qquad \text{.....(7)}$$

Now consider the integral

$$\int_{4N}^{\infty} \phi(t) \frac{dt}{t^3} = \sum_{n=N}^{\infty} \int_{4n}^{4n+4} \phi(t) \frac{dt}{t^3} \qquad \text{.....(8)}$$

$$= \sum_{n=N}^{\infty} \left\{ \int_{4n}^{4n+1} (t - 4n) \frac{dt}{t^3} + \int_{4n+1}^{4n+3} (2 - t + 4n) \frac{dt}{t^3} + \int_{4n+3}^{4n+4} (t - 4n - 4) \frac{dt}{t^3} \right\}.$$

Evaluating these integrals and simplifying, this becomes

$$\sum_{n=N}^{\infty} \left\{ \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{4n+1} - \frac{1}{4n+3}\right) \right\}$$

$$= \frac{1}{8N} - \sum_{n=2N}^{\infty} \frac{(-1)^n}{2n+1}$$

$$= \frac{1}{8N} - \sum_{n=4N+1}^{\infty} \frac{\chi(n)}{n}. \qquad \text{.....(9)}$$

Further, substituting (7) in (8), we have

$$\int_{4N}^{\infty}\phi(t)\frac{dt}{t^2}=2\left(\frac{2}{\pi}\right)^2\int_{4N}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\sin\frac{1}{2}\pi nt\right\}\frac{dt}{t^3}$$

$$=2\left(\frac{2}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}\int_{4N}^{\infty}\sin\frac{1}{2}\pi nt\,\frac{dt}{t^3}. \quad \text{.....(10)}$$

The inversion is justified by the absolute convergence of all the series and integrals.

Now, integrating by parts $(2k-2)$ times,

$$2\left(\frac{2}{\pi n}\right)^2\int_{4N}^{\infty}\sin\frac{1}{2}\pi nt\,\frac{dt}{t^3}=\frac{2!}{(2\pi nN)^3}-\frac{3!}{(2\pi nN)^5}+\dots$$

$$+(-1)^{k-1}\frac{(2k)!}{(2\pi nN)^{2k+1}}$$

$$+(-1)^k(2k+2)!\left(\frac{2}{\pi n}\right)^{2k+2}\int_{4N}^{\infty}\sin\frac{1}{2}\pi nt\,\frac{dt}{t^{2k+3}}.$$

Hence, by (9) and (10),

$$\frac{1}{8N}-\sum_{n=4N+1}^{\infty}\frac{\chi(n)}{n}=\frac{2!}{(2\pi N)^3}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^3}-\frac{3!}{(2\pi N)^5}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^5}+\dots$$

$$+(-1)^{k-1}\frac{(2k)!}{(2\pi N)^{2k+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2k+1}}-R_k, \quad \text{.....(11)}$$

where

$$R_k=(-1)^{k+1}\left(\frac{2}{\pi}\right)^{2k+2}(2k+2)!\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2k+2}}\int_{4N}^{\infty}\sin\frac{1}{2}\pi nt\,\frac{dt}{t^{2k+3}}$$

$$=(-1)^{k+1}\left(\frac{2}{\pi}\right)^{2k+2}(2k+2)!\sum_{n=1}^{\infty}\chi(n)\int_{4Nn}^{\infty}\sin\frac{1}{2}\pi u\,\frac{du}{u^{2k+3}}. \quad \text{.....(12)}$$

Now *

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2k+1}}=\frac{1}{2}(-1)^k\left(\frac{\pi}{2}\right)^{2k+1}\frac{E_{2k}}{(2k)!}.$$

Hence

$$(-1)^k\frac{(2k)!}{(2\pi N)^{2k+1}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{2k+1}}=\frac{E_{2k}}{2(4N)^{2k+1}}.$$

Further, $E_0=1$, so

$$\frac{1}{8N}=\frac{E_0}{2(4N)},$$

and (11) becomes

$$\sum_{n=4N+1}^{\infty}\frac{\chi(n)}{n}=\frac{1}{2}\sum_{r=0}^{k}\frac{E_{2r}}{(4N)^{2r+1}}+R_k.$$

Hence, by (1),

$$\frac{1}{4}\pi=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}=\sum_{n=1}^{4N}\frac{\chi(n)}{n}+\frac{1}{2}\sum_{r=0}^{k}\frac{E_{2r}}{(4N)^{2r+1}}+R_k,$$

and (2) follows immediately. It remains to prove the inequality (3). Now

* K. Knopp, *loc. cit.*, 240.

the integral in (12) can be regarded as the sum of an alternating series whose first term is less than

$$\frac{1}{(4Nn)^{2k+3}}\int_{4Nn}^{4Nn+2}\sin\frac{1}{2}\pi u\,du=\frac{4}{\pi(4Nn)^{2k+3}}.$$

Hence, by (12),

$$\begin{aligned}|R_k|&<\left(\frac{2}{\pi}\right)^{2k+2}(2k+2)!\,\frac{4}{\pi(4N)^{2k+3}}\sum_{n=1}^{\infty}\frac{|\chi(n)|}{n^{2k+3}}\\&=\frac{2(2k+2)!}{(2\pi N)^{2k+3}}\left\{1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^{2k+3}}\right\}\\&<\frac{2(2k+2)!}{(2\pi N)^{2k+3}}\left\{1+\int_0^{\infty}\frac{dt}{(2t+1)^{2k+3}}\right\}\\&=\frac{2(2k+2)!}{(2\pi N)^{2k+3}}\left\{1+\frac{1}{4k+4}\right\}\\&=\frac{(4k+5)(2k+1)!}{(2\pi N)^{2k+3}},\end{aligned}$$

thus proving (3).

4. *Numerical results.*

The Euler numbers can readily be calculated from the formulae * $E_0=1$, $E_{2n+1}=0$, and

$$E_{2n}+\binom{2n}{2}E_{2n-2}+\binom{2n}{4}E_{2n-4}+\ldots+\binom{2n}{2n}E_0=0.$$

We find that

$E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$, $E_{10}=-50521$, $E_{12}=2702765$, etc.

If we put $N=1$, $k=2$ in (2) and neglect R_k, we have

$$\frac{1}{4}\pi\doteq\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{4}-\frac{1}{4^3}+\frac{5}{4^5}\right),$$

whence $\pi\doteq 3{\cdot}14518$, *i.e.* an error of $3{\cdot}6\times10^{-3}$.

Further such results are tabulated below.

N.	k.	Computed value of π.	Error.
2	1	3·14133	$2{\cdot}6\times10^{-4}$
2	4	3·1415995	$6{\cdot}9\times10^{-6}$
4	1	3·1415835	$9{\cdot}1\times10^{-6}$
4	2	3·14159307	$4{\cdot}2\times10^{-7}$
4	4	3·1415926584	$4{\cdot}8\times10^{-9}$
4	6	3·14159265384	$2{\cdot}5\times10^{-10}$

The true value of π is 3·141592653589793, correct to 15 places of decimals.

5. *Further remarks.*

The formula (2) can also be regarded as a particular case of an analogue of the Euler-Maclaurin summation formula.† With appropriate conditions on

* K. Knopp, *loc. cit.*, 239.

† K. Knopp, *loc. cit.*, 520–40. We use the notation

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$$

for the Bernoulli numbers, as Knopp does. The formula (14) can be proved in the same way as (13).

$f(x)$ the corresponding forms of this formula and the analogue are

$$\frac{1}{2}f(N)+\sum_{n=N+1}^{\infty}f(n)\sim\int_{N}^{\infty}f(t)\,dt+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(N), \quad \ldots\ldots\ldots\ldots(13)$$

$$\sum_{n=4N}^{\infty}\chi(n)f(n)\sim\frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(2r)!}f^{(2r)}(4N), \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

as $N\to\infty$. If we put $f(x)=1/x$ in (14) we can derive (2).

The formula (2) can also be extended to the case when the series

$$\sum_{n=1}^{N}\frac{\chi(n)}{n}$$

is terminated after any finite number of terms.

We can use the method of Section 3 to show that

$$\frac{1}{4}\pi-\left\{\sum_{n=1}^{N-1}\frac{\chi(n)}{n}+\frac{1}{2}\frac{\chi(N)}{N}\right\}=P_N,$$

where, for N even,

$$P_N\sim\pm\frac{1}{2}\sum_{r=0}^{\infty}\frac{E_{2r}}{(N)^{2r+1}}$$

with positive or negative sign according as $N=4M$ or $N=4M+2$, and, for N odd,

$$P_N\sim\pm\frac{1}{2}\sum_{r=1}^{\infty}\frac{B_{2r}2^{2r}(2^{2r}-1)}{2r(N)^{2r}}$$

with positive or negative sign according as $N=4M+1$ or $N=4M-1$.

A. P. G.

1472. Probably he [Wordsworth] studied little beyond French and Italian during his Cambridge life; not, however, at any time forgetting (as I had so much reason to complain, when speaking of my Oxonian contemporaries) the literature of his own country. It is true that he took the regular degree of A.B., and in the regular course; but this was won in those days by a mere nominal examination, unless where the mathematical attainments of the student prompted his ambition to contest the splendid distinction of Senior Wrangler. This, in common with all other honours of the University, is won in our days with far severer effort than in that age of relaxed discipline; but at no period could it have been won, let the malicious say what they will, without an amount of mathematical skill very much beyond what has ever been exacted of its *alumni* by any other European University. Wordsworth was a profound admirer of the sublimer mathematics; at least of the higher geometry. The secret of this admiration for geometry lay in the antagonism between this world of bodiless abstraction and the world of passion. And here I may mention appropriately, and I hope without any breach of confidence, that, in a great philosophic poem of Wordsworth's, which is still in MS., and will remain in MS. until after his death, there is, at the opening of one of the books, a dream, which reaches the very *ne plus ultra* of sublimity, in my opinion, expressly framed to illustrate the eternity, and the independence of all social modes or fashions of existence, conceded to these two hemispheres, as it were, that compose the total world of human power—mathematics on the one hand, poetry on the other.—Thomas De Quincey, *Reminiscences of the English Lake Poets* (Everyman's Library Edition, page 124). [Per Dr. H. Lowery.]

THE LIMAÇON AND THE CARDIOID.

By J. P. McCarthy.

The polar equation to any curve can be written $r=f(\theta)$.

If P and Q are inverse points with respect to a circle whose centre O is at the pole and whose radius is k, then, since $OP \cdot OQ = k^2$, it follows that the polar equation to the inverse curve is $k^2/r = f(\theta)$.

It is proposed to find by inversion some properties of the limaçon and the cardioid that follow from certain well-known properties of the conics: others will suggest themselves. For convenience the same letter is used to denote both a point on the original figure and the corresponding point on the figure obtained by inversion.

I. *The Central Conic and the Limaçon.*

The polar equation to the conic is, with the usual notation, $l/r = 1 + e\cos\theta$, the focus S of the conic being the pole or origin. The equation to the inverse curve is, therefore, $lr = k^2(1 + e\cos\theta)$, which is a limaçon whose pole is at the origin S. We have the following pairs of theorems:

1. (*a*) If PSQ is a focal chord of a conic, the locus of the intersection of the tangents at P and Q is a straight line and each tangent subtends a right angle at the focus.

(*b*) If PSQ is a variable chord of a limaçon and if two circles through S touch the curve at P, Q respectively and intersect again at R, the locus of R is a circle passing through S and their common chord is perpendicular to PQ.

2. (*a*) If PSQ is a focal chord of a conic, then $1/SP + 1/SQ$ is constant and equal to $2/SL$.

(*b*) If PSQ is a chord of a limaçon, then $SP + SQ$, that is, PQ, is constant and equal to $2SL$.

3. (*a*) If TP, TQ are tangents to a conic, then the angles TSP, TSQ are equal.

(*b*) If two circles are drawn through S to touch a limaçon at P, Q respectively and intersect again at T, then the angles TSP, TSQ are equal.

4. (*a*) The foot of the perpendicular from the focus to a tangent to a conic lies on a fixed circle.

(*b*) A circle is drawn through S to touch a limaçon: the other extremity of the diameter through S lies on a fixed circle.

5. (*a*) PSQ is a focal chord of a conic: the circles through S, P and through S, Q orthogonal to the conic are drawn. The locus of their other intersection is a straight line.

(*b*) PSQ is a chord of a limaçon: the normals at P and Q intersect on a fixed circle through S.

6. (*a*) The tangents to a conic at the extremities of the latus rectum LL' intersect at X, the foot of the directrix.

(*b*) The two circles passing through S and touching the limaçon at L and L' respectively intersect on the axis (at X).

7. (*a*) The tangents to a conic at the ends of any chord passing through a fixed point intersect on a straight line.

(*b*) If any circle through S and a fixed point O cuts a limaçon at P and Q, the other intersection of the two circles which pass through S and touch the limaçon at P and Q respectively lies on a fixed circle.

II. *The Parabola and the Cardioid.*

The polar equation to the parabola is $l/r = 1 + \cos\theta$, the focus S being

the pole. The equation to the inverse curve is $lr = k^2(1 + \cos\theta)$, which is a cardioid having a cusp at the pole. We have the following pairs of theorems :

1. (*a*) The tangents at the extremities of a focal chord of a parabola intersect at right angles on a fixed straight line.

(*b*) PSQ is any chord of a cardioid through the pole S. The circles through S touching the cardioid at P, Q respectively are orthogonal and their other point of intersection lies on a fixed circle through S.

2. (*a*) If the tangent and the normal at a point P on a parabola meet the axis in T and G respectively, then $ST = SP = SG$.

(*b*) A circle through S touches a cardioid at P and cuts the axis again at T. The circle through S and P orthogonal to the other circle meets the axis again at G : then $ST = SP = SG$.

3. (*a*) The foot of the perpendicular drawn from the focus to a tangent to a parabola lies on the tangent at the vertex A.

(*b*) A circle is drawn through S to touch the cardioid. The locus of the extremity of the diameter through S is a circle passing through S and touching the cardioid at A.

4. (*a*) The tangents to a parabola at the extremities of the latus rectum LL' intersect at right angles at X, the foot of the directrix.

(*b*) The two circles passing through S and touching the cardioid at L and L' respectively intersect orthogonally on the axis (at X).

5. (*a*) PSQ is a focal chord of a parabola. The circles drawn through S to touch the parabola at P and Q are orthogonal.

(*b*) PSQ is any chord of a cardioid drawn through the pole. The tangents at P and Q intersect at right angles.

6. (*a*) If two parabolas having the same focus intersect at two points, the angles between them at these points are equal.

(*b*) If two cardioids having the same pole intersect at two other points the angles between them at these points are equal (in each case the angle is half the angle between the axes).

7. (*a*) Two parabolas having the same focus and their axes in opposite directions are orthogonal.

(*b*) Two cardioids having the same pole and their axes in opposite directions are orthogonal.

8. (*a*) If two parabolas, having the same focus S and the same axis, intersect at P and if PS cuts them again at Q and R, then $SP^2 = SQ \,.\, SR$.

(*b*) If two cardioids, having the same pole S and the same axis, intersect at P and if PS cuts them again at Q and R, then $SP^2 = SQ \,.\, SR$.

9. (*a*) The circle through the three points of intersection of three tangents to a parabola passes through the focus.

(*b*) If three circles are drawn through the pole S, each touching the cardioid, their other points of intersection are collinear.

10. (*a*) The tangents at the extremities Q, Q' of a focal chord QSQ' of a parabola meet (at right angles) at O. The diameter OPV cuts the curve at P and bisects QQ' at V. If the tangent at P meets the other two tangents at R, R' respectively, then the points O, R, V, S, R' lie on a circle.

(*b*) The circles drawn through the pole S of a cardioid to touch the curve at the extremities Q, Q' of a chord QSQ' through the pole intersect (orthogonally) at O. If the circle touching QQ' at S and the cardioid at P meets the other two circles in R and R', then R, O, R' lie on a circle meeting QQ' at V and O, P, V lie on a circle which touches the axis at S.

J. P. McC.

TWO TRIPOS QUESTIONS.

By G. N. Watson.

The theorem that, if $y^3 + 3xy + 2x^3 = 0$, then

$$x^2(1+x^3)\frac{d^2y}{dx^2} - \frac{3x}{2}\frac{dy}{dx} + y = 0$$

has been the topic of four recent notes in the *Gazette*, namely 1542 (not 1546) by E. H. N., 1610 by Mr. F. Bowman, 1684 by Mr. F. M. Goldner, and 1801 by Mr. H. V. Lowry. It seems worth while to supplement these notes by some comments which were for the most part suggested by the concluding lines of the note by E. H. N.

The writers of the four notes have looked at the theorem from the point of view of a candidate in an examination; if it were set in a Higher Certificate Distinction Paper, I have little doubt that a fair proportion of the candidates would recognise that the cubic is a unicursal cubic, they would write $x/y = u$ (or make some equivalent substitution), they would express x and y as rational functions of u, and they would realise that the question must inevitably come out by calculating accurately the differential coefficients of y with respect to x and then substituting in the left-hand side of the differential equation. In fact, they would reproduce the substance of the work of E. H. N. with the omission of his refinement of taking the reciprocal of y as an auxiliary variable to abbreviate the analysis.

Few candidates, I think, would attempt a solution on the lines of Note 1684, partly from a prevalent impression that the Tartaglia-Cardan solution of a cubic equation is not of much practical use and partly from a tendency to keep differential equations and theory of equations in separate water-tight compartments; and I doubt whether any candidate who realised that the cubic is unicursal would adopt the method of Note 1801.

There is, however, another and not less interesting aspect of the theorem, namely the method by which it was constructed; and I now proceed to set out the steps by which I believe the result was attained. It seems to me to be nothing but a well-disguised form of what may be described as a standard Higher Certificate question, namely the construction of the differential equations satisfied by such functions as

$$\{\sqrt{(z^2+1)} \pm z\}^m, \quad \cos(m \sin^{-1} z), \quad \sin(m \sin^{-1} z),$$

or functions obtainable from them by elementary transformations.

We are all familiar with the assertions that, if

$$w = \{\sqrt{(z^2+1)} \pm z\}^m,$$

with m any constant (not necessarily an integer), then

$$\frac{dw}{dz} = \pm\frac{mw}{\sqrt{(z^2+1)}},$$

and, by differentiating again,

$$\frac{d^2w}{dz^2} = \pm\frac{m}{\sqrt{(z^2+1)}}\frac{dw}{dz} \mp \frac{mwz}{(z^2+1)^{\frac{3}{2}}}$$

$$= \frac{m^2w}{z^2+1} - \frac{z}{z^2+1}\frac{dw}{dz},$$

so that

$$(z^2+1)\frac{d^2w}{dz^2} + z\frac{dw}{dz} - m^2w = 0.$$

The required result will now be obtained by straightforward changes of variables in this differential equation.

It is convenient to write Δ for the differential operator $z(d/dz)$; and it is clear that we have proved that, if w is any expression of the form

$$c_1\{\sqrt{(z^2+1)}+z\}^m + c_2\{\sqrt{(z^2+1)}-z\}^m,$$

with c_1 and c_2 arbitrary constants, then w is a solution of the differential equation

$$z^2(\Delta^2 - m^2)w + \Delta(\Delta - 1)w = 0.$$

We now specialise this result by taking $m = \frac{1}{3}$; and, since cube roots are now involved, it is natural to expect that some simplification will be effected by working with a new independent variable x in place of z, the relation between the variables being $z^2 = x^3$; and then, if ϑ denotes the differential operator $x(d/dx)$, we have $3\Delta = 2\vartheta$.

When we carry out these changes in notation, we see that, if w is any expression of the form

$$c_1\{\sqrt{(x^3+1)}+\sqrt{x^3}\}^{\frac{1}{3}} + c_2\{\sqrt{(x^3+1)}-\sqrt{x^3}\}^{\frac{1}{3}},$$

then w is a solution of the differential equation

$$x^3(4\vartheta^2 - 1)w + 2\vartheta(2\vartheta - 3)w = 0.$$

In terms of differentiations with respect to x, $\vartheta(\vartheta - 1)$ is a simpler operator than ϑ^2; and we can change the ϑ^2 in the leading term of the differential equation into $\vartheta(\vartheta - 1) + \frac{1}{4}$ by the introduction of a new dependent variable y in place of w, the relation between the variables being $w = yx^{-\frac{1}{2}}$. Applications of the symbolic theorem

$$f(\vartheta)\,.\,(yx^{-\frac{1}{2}}) = x^{-\frac{1}{2}}f(\vartheta - \tfrac{1}{2})y,$$

where $f(\vartheta)$ is any polynomial in ϑ, then give the following result:

If y is any expression of the form

$$c_1\{\sqrt{(x^3+1)}+\sqrt{x^3}\}^{\frac{1}{3}}\sqrt{x} + c_2\{\sqrt{(x^3+1)}-\sqrt{x^3}\}^{\frac{1}{3}}\sqrt{x},$$

then y is a solution of the differential equation

$$x^3(4\vartheta^2 - 4\vartheta)y + (2\vartheta - 1)(2\vartheta - 4)y = 0.$$

This differential equation immediately reduces to

$$x^2(1+x^3)\frac{d^2y}{dx^2} - \frac{3x}{2}\frac{dy}{dx} + y = 0.$$

We proceed to assign special values to c_1 and c_2 in the expression for y; but we first prescribe the values of the many-valued functions occurring in y. We select either branch of the functions $\sqrt{(x^3+1)}$ and $\sqrt{x}$, and we then define $\sqrt{x^3}$ to mean $(\sqrt{x})^3$; we next select any one of the three branches of

$$\{\sqrt{(x^3+1)}+\sqrt{x^3}\}^{\frac{1}{3}},$$

and we then define $\{\sqrt{(x^3+1)}-\sqrt{x^3}\}^{\frac{1}{3}}$ to mean the reciprocal of this function. When these choices have been made, we take c_2 to be any one of the three cube roots of unity, say ω, and lastly we take c_1 to be $-\omega^{-1}$. Since the three equations comprised in the formula

$$y = A\omega - B\omega^{-1}$$

are precisely equivalent to the single equation

$$y^3 + 3ABy + B^3 - A^3 = 0,$$

we see that, when we insert the appropriate values of A and B, and so obtain the equation

$$y^3 + 3xy + 2x^3 = 0,$$

we have arrived at the result that this equation possesses the property that each of its roots in y is a solution of the differential equation

$$x^2(1+x^3)\frac{d^2y}{dx^2} - \frac{3x}{2}\frac{dy}{dx} + y = 0,$$

and this is the result to be proved.

The reader who has gone so far will doubtless have wondered what grounds I had for asserting that the work just given was the method by which the result (which was set in the Tripos of 1903) was constructed. The answer to his enquiry is to be found in another problem, which was set in Part II of the Tripos of 1900. The link connecting the two problems is the fact that the late Professor Burnside examined on both occasions. The problem set in 1900 is as follows :

Prove that, if

$$y^5 - 5y^3 + 5y - 4x + 2 = 0,$$

then

$$\frac{d^2y}{dx^2} + \frac{2x-1}{2x(x-1)}\frac{dy}{dx} - \frac{y}{25x(x-1)} = 0.$$

To obtain this result (in which the differential equation is manifestly hypergeometric), revert to the theorem proved earlier in this paper, namely, that, if w is any expression of the form

$$c_1\{\sqrt{(z^2+1)} + z\}^m + c_2\{\sqrt{(z^2+1)} - z\}^m,$$

then w is a solution of the differential equation

$$z^2(\Delta^2 - m^2)w + \Delta(\Delta - 1)w = 0,$$

and make a change in the independent variable, the new variable x being defined in terms of z by the equation $z = i\sqrt{x}$; and then let ϑ denote the differential operator $x(d/dx)$, so that $\Delta = 2\vartheta$. There will be no risk of confusing this x and ϑ with the x and ϑ used in the previous problem.

We thus see that any expression of the form

$$c_1\{\sqrt{(1-x)} + i\sqrt{x}\}^m + c_2\{\sqrt{(1-x)} - i\sqrt{x}\}^m$$

is a solution of the differential equation

$$-x(4\vartheta^2 - m^2)w + 2\vartheta(2\vartheta - 1)w = 0 ;$$

and, when we specialise this result by taking $m = 2/5$, the differential equation becomes

$$x(1-x)\frac{d^2w}{dx^2} + (\tfrac{1}{2} - x)\frac{dw}{dx} + \frac{w}{25} = 0.$$

We now prescribe the values of the many-valued functions occurring in w. We select either branch of the functions $\sqrt{(1-x)}$ and $\sqrt{x}$, and then select any one of the five branches of

$$\{\sqrt{(1-x)} + i\sqrt{x}\}^{\frac{2}{5}} ;$$

we then define $\{\sqrt{(1-x)} - i\sqrt{x}\}^{\frac{2}{5}}$ to mean the reciprocal of this function. When these choices have been made, we take c_1 to be any one of the five fifth roots of unity, say ϵ, and lastly we take c_2 to be ϵ^{-1}.

We now have five values of w, each satisfying the differential equation, namely

$$w = \epsilon\{\sqrt{(1-x)} + i\sqrt{x}\}^{\frac{2}{5}} + \epsilon^{-1}\{\sqrt{(1-x)} - i\sqrt{x}\}^{\frac{2}{5}};$$

and, for these values of w, we have

$$\pm\sqrt{(w^2-4)} = \epsilon\{\sqrt{(1-x)} + i\sqrt{x}\}^{\frac{2}{5}} - \epsilon^{-1}\{\sqrt{(1-x)} - i\sqrt{x}\}^{\frac{2}{5}}.$$

Hence we have the two formulae :

$$w \pm \sqrt{(w^2-4)} = 2\epsilon\{\sqrt{(1-x)} + i\sqrt{x}\}^{\frac{2}{5}},$$
$$w \mp \sqrt{(w^2-4)} = 2\epsilon^{-1}\{\sqrt{(1-x)} - i\sqrt{x}\}^{\frac{2}{5}},$$

from which it immediately follows that

$$\{w + \sqrt{(w^2-4)}\}^5 + \{w - \sqrt{(w^2-4)}\}^5 = 32\{\sqrt{(1-x)} + i\sqrt{x}\}^2 + 32\{\sqrt{(1-x)} - i\sqrt{x}\}^2.$$

On simplifying this result, we see that the five values of w are the roots of the quintic equation

$$2w^5 + 20w^3(w^2-4) + 10w(w^2-4)^2 = 64(1-2x),$$

i.e.

$$w^5 - 5w^3 + 5w = 2 - 4x;$$

and, conversely, each root of this quintic equation is expressible in the form

$$c_1\{\sqrt{(1-x)} + i\sqrt{x}\}^{\frac{2}{5}} + c_2\{\sqrt{(1-x)} - i\sqrt{x}\}^{\frac{2}{5}},$$

and consequently satisfies the differential equation

$$x(1-x)\frac{d^2w}{dx^2} + (\tfrac{1}{2} - x)\frac{dw}{dx} + \frac{w}{25} = 0;$$

and this establishes the required result.

The foregoing analysis seems to me to provide reasonably convincing evidence that the two problems owe their origin to the same author ; and their family resemblance would have been even more conspicuous if I had not deliberately refrained from using the notation of hypergeometric functions in writing out the work. In fact, I imagine that the construction of the problems did not follow exactly the lines that I have laid out here, but started with the known result that each of the expressions

$$\{\sqrt{(1-x)} \pm i\sqrt{x}\}^m$$

satisfies the same hypergeometric equation as the hypergeometric series $F(\frac{1}{2}m, -\frac{1}{2}m ; \frac{1}{2} ; x)$, and that the problem of 1900 was then derived in the way that I have described ; and, when a problem of the same nature was required in 1903, the choice of a simpler value for m made it unnecessary to lay any stress on the hypergeometric character of the differential equation, and so led the way to the simplification effected by the changes in both independent and dependent variables.

It is, of course, possible to obtain numerous similar but more complicated results by choosing $1/m$ or $2/m$ to be an odd integer other than 3 or 5. It does not seem necessary to state any such results here, since the reader who is sufficiently interested should have no difficulty in constructing them for himself. G. N. W.

1473. Roughly, the length of its borders increases arithmetically while its area increases geometrically, when any country grows larger.—J. T. MacCurdy, *The Structure of Morale* (Cambridge, 1943), p. 89. [Per Mr. F. Puryer White.]

MATHEMATICAL NOTES.

1843. *On Notes* 1542, 1610, 1684, 1801.

Mr. Lowry has now led us back to the orthodox theory of the linear differential equation satisfied by an algebraic function, but some readers may not recognise the generality of the process, and having set the ball rolling in an attempt to amuse, perhaps I may bring it to rest by an attempt to instruct.

We have

$$y^3 + 3xy + 2x^3 = 0, \quad \text{.....(i)}$$

and therefore

$$dy/dx = -(y + 2x^2)/(y^2 + x). \quad \text{.....(ii)}$$

As functions of y, the cubic function $y^3 + 3xy + 2x^3$ and the quadratic function $y^2 + x$ have no common factor. Hence there is an identity

$$y + 2x^2 \equiv (y^3 + 3xy + 2x^3)\phi(y) - (y^2 + x)\psi(y),$$

where $\phi(y)$ is a polynomial in y of lower degree than $y^2 + x$, and $\psi(y)$ is a polynomial in y of lower degree than $y^3 + 3xy + 2x^3$; in these polynomials, the coefficients are functions of x. Writing $\phi(y) = Ay + B$ and adjusting the coefficients of y^4, y^3, y^2 in turn, we have $\psi(y) = Ay^2 + By + 2Ax$, whence

$$2Ax^3 + 2Bx = 1, \quad 2Bx^3 - 2Ax^2 = 2x^2.$$

Thus

$$2x(1 + x^3)(y + 2x^2) = -\{xy - (1 + 2x^3)\}(y^2 + 3xy + 2x^3) + \{xy^2 - (1 + 2x^3)y + 2x^2\}(y^2 + x),$$

always, as we may verify, and

$$2x(1 + x^3)(y + 2x^2) = \{xy^2 - (1 + 2x^3)y + 2x^2\}(y^2 + x)$$

whenever $y^3 + 3xy + 2x^3 = 0$. This is the origin of the equation

$$2x(1 + x^3)dy/dx = -xy^2 + (1 + 2x^3)y - 2x^2, \quad \text{.....(iii)}$$

which Mr. Lowry rightly regards as fundamental.

It is clear that if we differentiate (iii), substitute for dy/dx in the resulting equation from (iii) itself, and reduce powers of y as far as possible by repeated use of (i) in the form $y^3 = -3xy - 2x^3$, we must reach an equation of the form

$$X^2 d^2y/dx^2 = P_{22}y^2 + P_{21}y + P_{20}, \quad \text{.....(iv)}$$

where $X = 2x(1 + x^3)$ and P_{22}, P_{21}, P_{20} also are polynomials in x. Between (iii) and (iv) we can eliminate the term in y^2, obtaining the equation

$$xX^2\frac{d^2y}{dx^2} + XP_{22}\frac{dy}{dx} - \{xP_{21} + (1 + 2x^3)P_{22}\}y = x(P_{20} - 2xP_{22}).$$

Thus we have direct operations for constructing a linear differential equation of the second order satisfied by y;—what we do not know until we do the work in detail is that this linear equation is homogeneous.

Mr. Lowry is to be congratulated on his opportunism in reducing the equation derived from (iii). He recognises that the only unwelcome term arising from the differentiation is $-2xy\,dy/dx$, and that because (ii) can be replaced by

$$dy/dx = (y^2 + 2x^2y)/2x(y + x^2), \quad \text{.....(v)}$$

this term can be expressed immediately in a suitable form; this effective alternative to (ii) exists only because there is no multiple of y^2 in (i). We

can use (v) alone, instead of combining (v) with (ii) as Mr. Lowry does, to find (iii). Writing, as must be possible,

$$y^2 + 2x^2y \equiv C(y^3 + 3xy + 2x^3) - (Cy^2 + Dy + 2Cx)(y + x^2),$$

we have

$$Cx^2 + D = -1, \quad Cx - Dx^2 = 2x^2,$$

and the identity which leads from (v) to (iii) is

$$(1 + x^3)(y^2 + 2x^2y) = x(y^3 + 3xy + 2x^3) - \{xy^2 - (1 + 2x^3)y + 2x^4\}(y + x^2).$$

The outlines of a general theory should now be visible. If y is an algebraic function of x defined by an irreducible equation $F(y, x) = 0$ of degree n in y, then $dy/dx = -F_x/F_y$, where F_y, as a function of y, is a polynomial of degree $n - 1$. Polynomials $\phi(y)$, $\psi_1(y)$ in y, with coefficients rational in x, are determined uniquely by the conditions that $F_x \equiv F\phi - F_y\psi_1$ identically and that the degree of ψ_1 in y is less than n. When $\phi = 0$, $F_x = -F_y\psi_1$, that is $dy/dx = \psi_1$. By the process already outlined we can derive in succession polynomials ψ_2, ψ_3, ... in y, each of degree less than n, such that $d^my/dx^m = \psi_m$. The function y satisfies as a rule a linear differential equation of order $n - 1$ that is not homogeneous, and a homogeneous linear differential equation of order n, for we usually need $n - 1$ equations to eliminate linearly the $n - 2$ powers y^2, y^3, ... , y^{n-1}, and one more equation if the terms independent of y are to disappear also. E. H. N.

1844. *A mathematical theorem about golf.*

It is usually held that steadiness, as against brilliancy, will tell more by strokes than by holes. It should follow that, if A is the steadier of two nearly equal players, B the more brilliant, and they are equal over a series of medal rounds, then B should have the advantage in a series of matches by holes. This seems to be the commonly accepted doctrine, but the evidence is inconclusive, since matches by holes between any given pair of leading golfers are comparatively rare. It may therefore be worth while to point out that a little mathematical analysis points, for what it is worth, to the opposite conclusion.

I construct a mathematical model as follows. Suppose that the course consists of 18 holes, all par fours (an immaterial simplification). Suppose that A is a completely mechanical player who does every hole in four (so that the conditions are similar to those of a bogey competition); and that B has equal chances x per stroke of making a *supershot* which gains a stroke or a *subshot* which costs one. It is plain that x cannot exceed $\frac{1}{2}$, and will be fairly small in any case at all corresponding to reality; that B will average par; and that the players will be equal over a series of medal rounds. According to the apparently accepted doctrine, B should win a long series of matches by holes.

To see the working of the model, suppose first that we may neglect terms of order x^2, *i.e.* consequences of B's playing more than one abnormal shot at the same hole. Then, to do a three, B must produce a supershot at one of his first *three* strokes, while he will take five if he makes a subshot at one of his first *four*. He will thus have a net expectation $4x - 3x$ or x of loss on the hole, and should lose the match, contrary to common expectation.

This approximation is too rough, but there is no difficulty in calculating the chances exactly. I find that B's chance of winning a hole is $3x - 9x^2 + 10x^3$, and his chance of losing $4x - 18x^2 + 40x^3 - 35x^4$, so that there is a balance

$$f(x) = x - 9x^2 + 30x^3 - 35x^4$$

against him. Mr. A. M. Binnie has plotted $f(x)$ for me. It increases to a

maximum, about ·037, for x about ·09; and then decreases, vanishing for x about ·37 and falling to $-\frac{3}{16}$ for $x=\frac{1}{2}$. There are two inflexions, for x about ·16 and ·27, but these merely flatten part of the curve and do not affect its appearance seriously. The model has, of course, lost all its plausibility by the time that x is near $\frac{1}{2}$; a player with $x=\frac{1}{2}$ would be a "completely erratic" player, *all* of whose strokes are abnormal, and the conclusion that such a player would win, paradoxical as it may seem, need not disturb us. But for smaller x, and in particular for x about $\frac{1}{16}$, the model seems fairly reasonable.

If experience points the other way—and I cannot deny it, since I am no golfer—what is the explanation? I asked Mr. Bernard Darwin, who should be as good a judge as one could find, and he put his finger at once on a likely flaw in the model. To play a "subshot" is to give yourself an *opportunity* of a "supershot" which a more mechanical player would miss: if you get into a bunker you have an opportunity of recovering without loss, and one which you are naturally keyed up to take. Thus the less mechanical player's chance of a supershot is to some extent automatically increased. How far this may resolve the paradox, if it is one, I cannot say, and changes in the model make it unpleasantly complex.

I add one word about possible testing of the model. There are few data about matches between, say, Hagen and R. T. Jones. But the model, as I said, is more like that of a bogey competition, and there must be abundant information about them. If a player averages bogey, does he win or lose when he plays against bogey by holes? There should be plenty of data for testing that.

G. H. HARDY.

1845. *Find the penny.*

An ingenious puzzle, popular at the moment, runs as follows. There is one bad penny in a set of twelve; find which it is, and whether it is underweight or overweight, by not more than three weighings on a common balance. Those who have met this puzzle may be interested in the following generalisation:

If one amongst $\frac{1}{2}(3^n-2n+3)$ pennies is bad, the bad penny and its fault may be determined by n weighings ($n\geqslant 3$).

For instance, from 38 pennies the bad one can be detected in *four* weighings, and from 118 in *five* weighings.

The problem is essentially one of finding the solution of a number of inequalities. We shall denote the pennies by capital letters (with or without subscripts) and a group $X_m, X_{m+1}, \dots X_n$ will be denoted by $X(m, n)$; similarly $X(m, n)Y(p, q)$ denotes all the pennies X_r, Y_s, $m\leqslant r\leqslant n$, $p\leqslant s\leqslant q$. If all the pennies $X(m, n)$ are good, we write $X(m, n)=0$; if one of the group is overweight, we write $X(m, n)>0$, and if one is underweight $X(m, n)<0$. If two groups have equal weight we write $X(m, n)=Y(m, n)$, and if the first is heavier than the second, $X(m, n)>Y(m, n)$. The symbol $X(m, n)$ operates as a sum $X_m+X_{m+1}+\dots+X_n$. Since there is only *one* bad penny the equation $X(m, n)=Y(m, n)$ implies $X_r=Y_r=0$, $m\leqslant r\leqslant n$. If $X(m, n)$ is weighed against $Y(m, n)$ we write $X(m, n)\,|\,Y(m, n)$.

The theorem to be proved is a consequence of the following three lemmas.

Lemma I. If $A(1, 3^n)>0$, then the heavy penny can be determined in n weighings. For if $A(1, 3)>0$, then $A_2\,|\,A_3$ decides the issue, since

$$A_2=A_3\rightarrow A_1>0,\quad A_2>A_3\rightarrow A_2>0\quad\text{and}\quad A_2<A_3\rightarrow A_3>0.$$

And if the lemma is true for some value of n, say $n=m$, we deduce its truth for $n=m+1$; for 3^{m+1} pennies may be divided into 3 groups, G_1, G_2, G_3, each containing 3^m pennies, and

$$G_1=G_2\rightarrow G_3>0,\quad G_1>G_2\rightarrow G_1>0,\quad G_1<G_2\rightarrow G_2>0,$$

so that after $G_1 \mid G_2$ the issue is decided in a further m weighings, by hypothesis, making $m+1$ weighings in all. Whence, by induction, the lemma is true for any n.

Similarly, if $A(1, 3^n) < 0$, the light penny can be found by n weighings.

Lemma II. If $\kappa(n) = (3^n - 1)/2$, and if

$$A(1, \kappa(n)) > B(1, \kappa(n)), \quad C(1, 3^{n-1}) = 0,$$

then the bad penny and its fault can be found by n weighings.

For if $A_1 > B_1$ and $C_1 = 0$, then $A_1 \mid C_1$ decides the issue (in fact, $A_1 = C_1 \to B_1 < 0$, $A_1 > C_1 \to A_1 > 0$), which proves the case $n = 1$.

We derive the case $n = m+1$ from the case $n = m$. We observe first that if $A(1, \kappa(m+1)) > B(1, \kappa(m+1))$, then $A_r \geqslant 0$, $B_r \leqslant 0$, $1 \leqslant r \leqslant \kappa(m+1)$; consider now (noting $3^m + \kappa(m) = \kappa(m+1)$) the consequences of

$$A(1, 3^m) B(3^m + 1, 3^m + \kappa(m)) \mid C(1, 3^m) A(3^m + 1, 3^m + \kappa(m)).$$

If these groups are equal, then $B(1, 3^m) < 0$, and the issue is decided in a further m weighings, by Lemma I. If that on the left is heavier, then

$$A(1, 3^m) > 0$$

(since $C(1, 3^m) A(3^m + 1, 3^m + \kappa(m)) \geqslant 0$ and $B(3^m + 1, 3^m + \kappa(m)) \leqslant 0$); and, again by Lemma I, the issue is decided in a further m weighings.

If the group on the left is lighter, then

$$B(3^m + 1, 3^m + \kappa(m)) < A(3^m + 1, 3^m + \kappa(m)),$$

for $A(1, 3^m) \geqslant 0$, $C(1, 3^m) = 0$, and since each group in the inequality contains $\kappa(m)$ pennies, the issue is decided by m more weighings, by hypothesis. Whence, by induction, Lemma II is true for all n.

Lemma III. If $A(1, 4) \gtrless 0$ and $B(1, 3) = 0$, then just two weighings determine the bad penny and its fault. For $A(1, 3) \mid B(1, 3)$ followed by $A_2 A_4 \mid B_1 A_3$ decide the issue, since $A(1, 3) = B(1, 3) \to A_1 = A_2 = A_3 = 0$ and $A_4 \gtrless 0$ according as $A_2 A_4 \gtrless B_1 A_3$, and $A(1, 3) > B(1, 3) \to A_4 = 0$, and $A_1 > 0$, or $A_2 > 0$, or $A_3 > 0$ according as $A_2 A_4 \lessgtr B_1 A_3$, etc.

(Lemma III is a particular case of the general theorem, with $n = 2$, but requires the additional assumption $B(1, 3) = 0$).

Theorem I. n weighings suffice to determine the bad penny and its fault from a group of $p(n) = \frac{1}{2}(3^n - 2n + 3)$ pennies ($n \geqslant 3$).

Consider first the case $n = 3$, when there are twelve pennies.

Let the pennies be named $A(1, 4)$, $B(1, 4)$, $C(1, 4)$. Then $A(1, 4) \mid B(1, 4)$ decides the issue; for equality implies $C(1, 4) \gtrless 0$ and $A(1, 4) = B(1, 4) = 0$, so that a further two weighings suffice by Lemma III, and if $A(1, 4) \gtrless B(1, 4)$, then two further weighings suffice by Lemma II.

Assume the truth of the theorem for $n = m \geqslant 3$, and for the case $n = m+1$ let the pennies be $A(1, \kappa(m))$, $B(1, \kappa(m))$, $C(1, p(m))$. (Note that

$$2\kappa(m) + p(m) = 3^m - 1 + \tfrac{1}{2}(3^m - 2m + 3) = \tfrac{1}{2}(3^{m+1} - 2(m+1) + 3) = p(m+1)).$$

Then $A(1, \kappa(m)) \mid B(1, \kappa(m))$ decides the issue; for equality implies

$$C(1, p(m)) \gtrless 0,$$

whence the bad penny and its fault is detected by a further m weighings, by hypothesis. And if $A(1, \kappa(m)) \gtrless B(1, \kappa(m))$, then m more weighings suffice, by Lemma II. Thus $m+1$ weighings in all suffice. By induction, Theorem I is true for all values of n.

Theorem I is readily extended to cover a group of any number of pennies. To this end we prove extensions of the first two lemmas.

Lemma I.* If $A(1, n)>0$ then the bad penny and its fault can be detected by $q(n)$ weighings, where $q(n)$ is the least power of 3 not less than n. For $n=1$ the result is evident; if it holds for $n=1, 2, 3, \ldots, m-1$, then, for the case $n=m$, let 3μ be the first multiple of 3 not less than n, and consider $A(1, \mu) \mid A(\mu+1, 2\mu)$. If $A(1, \mu)>A(\mu+1, 2\mu)$, then $A(1, \mu)>0$, and since $3^{q(m)} \geqslant m$, so that $3^{q(m)-1} \geqslant \mu$, the issue is decided in a further $q(m)-1$ weighings, by hypothesis. Similarly, if $A(1, \mu)<A(\mu+1, 2\mu)$, $q(m)-1$ further weighings suffice. And if $A(1, \mu)=A(\mu+1, 2\mu)$, then $A(2\mu+1, m)>0$ and $q(m-2\mu)$ further weighings suffice; but $3^{q(m)-1} \geqslant \mu \geqslant m-2\mu$, so that $q(m-2\mu) \leqslant q(m)-1$, and the lemma is established for $n=m$. Whence by induction it holds for all n.

Lemma II*. If $A(1, n)>B(1, n)$ and $C(1, \theta_n)=0$, where $\theta_n=n-\kappa(\nu_n)$, and ν_n is the greatest integer such that $\kappa(\nu_n)<n$, then the bad penny and its fault is detected by ν_n+1 weighings.

We consider the consequences of

$$A(1, \theta_n)B(\theta_n+1, \theta_n+\kappa(\nu_n)) \mid C(1, \theta_n)A(\theta_n+1, \theta_n+\kappa(\nu_n)). \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Equality implies $B(1, \theta_n)<0$ and the issue is decided in a further $q(\theta_n)$ weighings. Now $\kappa(\nu_n+1) \geqslant n > \kappa(\nu_n)$ and $\kappa(\nu_n+1)=\kappa(\nu_n)+3^{\nu_n}$, so that $\theta_n \leqslant 3^{\nu_n}$; but $q(\theta_n)$ is the least power of 3 which is not less than θ_n, and so $q(\theta_n) \leqslant \nu_n$. Accordingly at most ν_n+1 weighings in all decide the issue.

If the left-hand side of (i) is the greater, then $A(1, \theta_n)>0$, and as before the issue is decided in a further ν_n steps. If the left-hand side of (i) is the lesser, then

$$B(\theta_n+1, \theta_n+\kappa(\nu_n)) < A(\theta_n+1, \theta_n+\kappa(\nu_n)),$$

and by Lemma II the issue is decided by a further ν_n weighings, which completes the proof of Lemma II*.

Theorem I*. If $p(n)<N \leqslant p(n+1)$, then $n+1$ weighings suffice to determine the bad penny and its fault from a group of N pennies ($n \geqslant 2$).

Consider first the cases $N=5, 6, 7, 8, 9, 10, 11$, where $N<p(3)$. In the cases 5, 6, 7, we consider $A(1, 2) \mid B(1, 2)$; equality implies $C(1, 2, 3) \gtrless 0$ (C_2 and/or C_3 may be absent), and the issue is decided by two more weighings (at most). If $A(1, 2) \gtrless B(1, 2)$, then $A_1B_2 \mid C_1A_2$ leads to a decision after one more weighing. In the cases 8, 9, 10, $A(1, 3) \mid B(1, 3)$ followed by $A(1, 2)B_3 \mid C(1, 2)A_3$ leads to a decision (using Lemma III). For case 11, $A(1, 4) \mid B(1, 4)$ followed by $B(1, 3)B_4 \mid C(1, 3)A_4$ leads to a decision.

We derive the theorem for $n=m$ from the case $n=m-1$.

We have $p(m)<N \leqslant p(m+1)$; if $N-p(m)=2\alpha$, then we group the pennies $A(1, \alpha)$, $B(1, \alpha)$, $C(1, p(m))$. Since $2\kappa(m)+p(m)=p(m+1) \geqslant N$, therefore $\alpha \leqslant \kappa(m)$.

Consider $A(1, \alpha) \mid B(1, \alpha)$; equality implies $C(1, p(m)) \gtrless 0$ and the issue is decided in a further m weighings, by Theorem I.

If $A(1, \alpha) \gtrless B(1, \alpha)$, then by Lemma II* a further $\nu_\alpha+1$ weighings suffice; but $\kappa(\nu_\alpha)<\alpha \leqslant \kappa(m)$, so that $\nu_\alpha<m$ and therefore $\nu_\alpha+1 \leqslant m$.

If $N-p(m)=2\alpha-1$, then we group the pennies

$$A(1, \alpha), \quad B(1, \alpha), \quad C(1, p(m)-1).$$

Since $N-p(m)$ is odd and $p(m+1)-p(m)=2\kappa(m)$, therefore $p(m+1)>N$, and so $\alpha \leqslant \kappa(m)$. As before, we consider $A(1, \alpha) \mid B(1, \alpha)$; equality implies $C(1, p(m)-1) \gtrless 0$, and by hypothesis m more weighings suffice.

If $A(1, \alpha) \gtrless B(1, \alpha)$, then $\nu_\alpha+1$ more weighings suffice and $\nu_\alpha+1 \leqslant m$.

Thus Theorem I* is proved for $n=m$, and by induction it is true for all n.

R. L. G.

1846. *Solution of linear differential equations with constant coefficients.*

1. Writing $D^r y$ for $d^r y/dx^r$ for ease of printing, let the differential equation be

$$D^n y + a_1 D^{n-1} y + \ldots + a_n y = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

under initial conditions

$$y = b_0,\ Dy = b_1,\ \ldots,\ D^{n-1} y = b_{n-1},\ \text{when } x = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

(The case in which the initial value of x is a non-zero constant may be similarly treated.)

It is assumed that the polynomial expression

$$E = p^n + a_1 p^{n-1} + \ldots + a_n$$

can be factorised. With any factor $p - \alpha$, write

$$E = (p - \alpha)(p^{n-1} + q_1 p^{n-2} + q_2 p^{n-2} + \ldots + q_{n-1}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

It follows at once that (i) may be written

$$DU - \alpha U = f(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

where

$$U = D^{n-1} y + q_1 D^{n-2} y + \ldots + q_{n-1} y.$$

The solution of (iv) is

$$U = e^{\alpha x} \int_0^x e^{-\alpha x} f(x)\,dx + U_0 e^{\alpha x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

where U_0 is the value of U when $x = 0$.

The process is now repeated with other factors of E. It is convenient to remove repeated factors at the end of the process, but the various forms which can arise require no separate consideration.

With a proper scheme of manipulation, the above method, over a number of years, has been found to be at least as rapid as more usual arrangements, and does not need the consideration of special cases.

2. However, the result for the Particular Integral is clumsy in appearance. The following modification due to Rayleigh * is more elegant (see § 3).

Put $f(x) = 0$ in the above working throughout. The complementary function is linear in $b_0, b_1, \ldots, b_{n-1}$. Let the term in b_{n-1} be

$$b_{n-1}\phi(x) \quad \text{or} \quad (D^{n-1} y)_0 \,.\, \phi(x). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}$$

Let $f(x)$ act from $x = x'$ to $x' + dx'$ and then cease. The effect of this is to cause an increment in $D^{n-1} y$ of $f(x')\,dx'$, which at x will give rise to

$$f(x')\,dx' \,.\, \phi(x - x'), \quad \text{by (vi)}$$

Hence the Particular Integral becomes

$$\int_0^x f(x')\phi(x - x')\,dx'.$$

3. Rayleigh (*loc. cit.*) deals with the second-order equation as follows:

The solution of

$$\ddot{u} + \kappa\dot{u} + n^2 u = 0.$$

with initial values $u = u_0$, $\dot{u} = \dot{u}_0$ at $t = 0$, is given as

$$ue^{-\frac{1}{2}\kappa t}\{(\dot{u}_0/n')\sin n't + \text{term in } u_0\},$$

where $n' = \sqrt{(n^2 - \frac{1}{4}\kappa^2)}$, and proceeds:

These results may be employed to obtain the solution of the complete equation

$$\ddot{u} + \kappa\dot{u} + n^2 u = U, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(4)}$$

* *Theory of Sound*, Vol. I (2nd edition, 1894), p. 74.

where U is an explicit function of the time : for from (2) [the solution given above] we see that the effect at time t of a velocity $\delta\dot{u}$ communicated at time t' is

$$u = \delta\dot{u} \,.\, e^{-\frac{1}{2}\kappa(t-t')} \sin n'\,(t-t')/n'.$$

The effect of U is to generate in time dt' a velocity $U dt'$ whose result at time t will therefore be

$$u = \frac{1}{n'}\, U\, dt' e^{-\frac{1}{2}\kappa(t-t')} \sin n'\,(t-t'),$$

and thus the solution of (4) is

$$u = \frac{1}{n'} \int^{t} e^{-\frac{1}{2}\kappa(t-t')} \sin n'(t-t') \,.\, U \,.\, dt',$$

U being the force at time t'. C. E. WRIGHT.

1847. *Cycles.*

Mr. Joint's illuminating Note 1767 on cycles prompts me to expand my original Note 1581. My interest in cycles was started by the discovery of the 5-cycle $u_{n+1}u_{n-1} = a(u_n + a)$.

I came across this in a strange way ; in finding three integers such that the sum or difference of any two of them is a square. My first solution was improved by my friend Mr. D. F. Ferguson, who produced a set of five solutions, of which mine was one, all of which depended on a single solution of a diophantine equation :

$$a^4 - c^2 = d^2(b^4 - c^2).$$

In examining these solutions I noticed that one could be derived from another by a certain substitution, and I thought at first that by repeating this process I could get an infinite set of solutions in the familiar diophantine way. However, the substitution upon being repeated four times gave the first solution again. In its essence the substitution amounted to the use of the recurrence formula $u_{n+1}u_{n-1} = u_n + 1$.

In the attempt to get an explanation of this 5-cycle I constructed other recurrence formulae which gave cycles. The simplest formulae for an m-cycle are linear

$$u_{m+n} - u_n = 0, \quad \text{which is trivial,}$$

and

$$\sum_{1}^{m} u_{r+n} = p,$$

the general solution of the second being

$$u_n = p/m + \sum_{1}^{m-1} A_r \alpha^n,$$

where $\alpha = \text{cis}\,(2\pi/m)$ and the $(m-1)$ arbitrary constants A_r are involved.

Consider now a relation of the second degree,

$$d w_{n+1} w_{n-1} = w_n(w_{n+1} + a w_n + b w_{n-1}).$$

Dividing by $w_n w_{n-1}$ and putting $w_{n+1}/w_n = v_{n+1}$, we have

$$d v_{n+1} = v_n v_{n+1} + a v_n + b.$$

Now put $v_{n+1} = (u_{n+1}/u_n) + d$; we get

$$u_{n+1} + (a+d)u_n + (ad+b)u_{n-1} = 0.$$

Hence if the equation

$$x^2 + (a+d)x + (ad+b) = 0$$

has roots of the form $r \operatorname{cis}(\pm l\pi/m)$, where l is prime to m, then

$$u_t = r^t \{A \cos(tl\pi/m) + B \sin(tl\pi/m)\},$$

so that $u_{t+m} = (-)^l r^m u_t$, and

$$\begin{aligned} v_{m+t+1} - d &= u_{m+t+1}/u_{m+t} \\ &= u_{t+1}/u_t \\ &= v_{t+1} - d\,; \end{aligned}$$

in other words, v_n and w_{n+1}/w_n are periodic. For the quadratic equation to have such roots, we must have

$$a + d = 2\cos(l\pi/m) \cdot \sqrt{(ad+b)}.$$

As an illustration, take $a=3$, $d=1$, $b=5$. Then $m=4$ and $w_{n+5}/w_{n+4} = w_{n+1}/w_n$; or, a 4-cycle is given by

$$v_{n+1} = v_n v_{n+1} + 3v_n + 5.$$

So far we have arranged that w_{n+1}/w_n is to be periodic. I cannot find the condition that w_n should be periodic. Obviously

$$\prod_1^m v_{n+r} = 1,$$

but what further relation must exist between a, b, d, m to ensure this? I suspect that $b=1$ is a necessary condition but that it is not sufficient is illustrated by $a=8/7$, $d=-5/7$, $b=1$, $m=3$, which gives a 3-cycle for v but not for w.

If $dw_{n+1}w_{n-1} = w_n(cw_{n+1} + aw_n + bw_{n-1})$ gives an m-cycle it is easy to show that the interchange of a, d and/or b, c will still give an m-cycle, if m is even. For example, since

$$w_{n+1}w_{n-1} = w_n(w_{n+1} + 2w_n + w_{n-1}) \text{ gives a 6-cycle}$$

it follows that

$$2w_{n+1}w_{n-1} = w_n(w_{n+1} + w_n + w_{n-1}) \text{ also gives a 6-cycle.}$$

The question now arises, why does the addition of p to the left-hand side destroy the second of these cycles but not the first? The answer is provided by Mr. Joint's analysis. I found that it was so by accident. When I was checking my results numerically I made a mistake in changing the subject of the formula and in spite of this I still got the cycle.

An interesting application of cycles can be illustrated by considering again the 4-cycle

$$v_{n+1} = v_n v_{n+1} + 3v_n + 5.$$

The terms of the cycle are

$$x, \quad (3x+5)/(1-x), \quad -(x+5)/(1+x), \quad (x-5)/(3+x)\,;$$

it follows that if α is a root of the equation

$$x(x+5)(x-5)(3x+5) - k(x+1)(x-1)(x+3) = 0,$$

then so is $(3\alpha+5)/(1-\alpha)$.

Returning now to the 5-cycle $u_{n+1}u_{n-1} = u_n + 1$, the terms of which are

$$x, \quad y, \quad (y+1)/x, \quad (x+y+1)/xy, \quad (x+1)/y,$$

it follows that if the point (a, b) lies on the cubic curve

$$(x+1)(y+1)(x+y+1) = kxy,$$

then the point $\{b, (b+1)/a\}$ will also lie on the curve.

From this we have a theorem about cubic curves :

A cubic curve touches AB at A, BC at B, CD at C and DA at D ; P_1 is any point on the curve. P_1A cuts the curve at P_2, P_2C cuts it at P_3, and in general $P_{2r-1}A$ cuts the curve in P_{2r} and $P_{2r}C$ cuts it at P_{2r+1}. Then P_m is the same point as P_n if $m \equiv n \pmod{10}$.

Mr. H. Todd of Bristol University has shown that when the cubic is transformed, the five points have elliptic parameters differing by one-fifth of a period. The clue to the periodicity of the cycle seems to lie in elliptic functions, and not as in the other cases in the circular functions. Mr. Todd has also discovered that

$$u_{n+3}u_n = u_{n+2} + u_{n+1} + 1$$

gives an 8-cycle.

We are still awaiting an interesting 7-cycle. R. C. LYNESS.

1848. *Approximation to the roots of an equation.*

The formula for nth roots given by Dr. H. W. Richmond in the *Gazette* (February, 1944) can be obtained by applying Newton's rule to the equation $x^n = R$ when this is written in the form

$$x^{\frac{1}{2}(n+1)} = R/x^{\frac{1}{2}(n-1)}.$$

The reason this form of the equation gives a better approximation is as follows. Let $f(x) = 0$ have a root near $x = a$: then a closer approximation is

$$a - \frac{1}{f'(a)} f(a) - \frac{f''(a)}{2\{f'(a)\}^3} \{f(a)\}^2 - \dots .$$

In some equations it is possible to multiply by a function with an arbitrary parameter, such as x^p, and then to choose the parameter p so that $f''(a)$ is of the same order as $f(a)$; this makes the last term above of the same order as the next term in the series, and so makes $a - f(a)/f'(a)$ correct to the third power of $f(a)$.

Thus, if

$$f(x) = x^{n+p} - Rx^p,$$

$$f''(x) = p(p-1)x^{p-2}(x^n - R) + 2px^{p-1}nx^{n-1} + n(n-1)x^p \,.\, x^{n-2}.$$

For $f''(x)$ to be of the same order as $f(x)$ it must vanish when $x^n = R$, and so $p = -\frac{1}{2}(n-1)$. Therefore the best approximation to the roots of $x^n = R$ is got by applying Newton's rule to

$$x^{\frac{1}{2}(n+1)} = R/x^{\frac{1}{2}(n-1)}.$$

Another case when this method can be applied is that of the equation $\sin x = x \cos x$. Multiplying by x^p, the second derivative of $x^p(\sin x - x\cos x)$ is

$$p(p-1)x^{p-2}(\sin x - x\cos x) + 2px^{p-1}x\sin x + x^p(x\cos x + \sin x),$$

which vanishes when $\sin x = x\cos x$ if $p = -1$. Hence the best results are obtained by applying Newton's rule to the equation written in the form

$$x^{-1}\sin x = \cos x.$$

It is possible to generalise the method to functions satisfying a given differential equation of the second order. Let the equation to be solved be $F(x) = 0$, where $F(x)$ satisfies the differential equation

$$F'' + PF' + QF = 0,$$

P and Q being functions of x. Let Newton's rule be applied to the equation in the form

$$f(x) \equiv \phi(x)F(x) = 0,$$

and let us try to choose $\phi(x)$ so that $f''(x)$ is of the same order as $f(x)$.

$$f''(x) = \phi''(x)F(x) + 2\phi'(x)F'(x) + \phi(x)F''(x)$$
$$= 2\phi'(x)F'(x) + \phi(x)F''(x) \text{ at } F(x) = 0.$$

But at $F(x) = 0$, $\quad F''(x) + PF'(x) = 0$.

Comparing these two equations,

$$2\phi'(x)/P = \phi(x),$$

whence

$$\phi(x) = \exp\left(\tfrac{1}{2}\int P\,dx\right).$$

If, for instance, $F(x) = J_0(x)$, $\phi(x) = \sqrt{x}$, so that the best closer approximation to a root a of $J_0(x) = 0$ is $a - h$, where

$$h = \{\sqrt{a}\,.\,J_0(a)\}/\{\sqrt{a}\,.\,J_0'(a) + \tfrac{1}{2}J_0(a)/\sqrt{a}\}$$
$$= J_0(a)/\{J_0'(a) + J_0(a)/a\}.$$

Taking $a = 2{\cdot}5$ this gives $2{\cdot}4045$, which is only 1 in 8000 out.

The corresponding formula for a closer approximation to the root of $F(x) = 0$ near to a is

$$a - \frac{F(a)}{F'(a) + \tfrac{1}{2}PF(a)}.$$

H. V. Lowry.

1849. *On Notes* 1691, 1692.

I disagree with the idea that formulae in spherical trigonometry should be proved geometrically: any figure drawn is a special case and does not give a general proof. For the proof of the fundamental formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

there are 8 cases: (1) b, c both acute, A any angle; (2) one of b, c acute, the other obtuse, A any angle; (3) b, c both obtuse, A any angle; (4) one of b, c a right angle, the other acute, A not $90°$; (5) one of b, c a right angle, the other obtuse, A not $90°$; (6) and (7), cases (4) and (5) but with A equal to $90°$; (8), b, c both right angles, A any angle. So to avoid a multiplicity of cases for any formula, the best way is to derive all formulae from this fundamental one.

References being to Todhunter and Leathem, *Spherical Trigonometry*, $\sin A \sin b = \sin B \sin a$ is proved as in Art. 45, then the half-angle results, $\sin \tfrac{1}{2}A$, etc., as in Art. 50, Napier's Analogies as in Art. 62, avoiding the far from obvious manipulation in Art. 61, Delambre's Analogies as in Art. 63. The proof of one of Delambre's Analogies in Note 1692 is far from attractive; to begin with, it is necessary to know $\cos^2 \tfrac{1}{2}(a+b)/\cos^2 \tfrac{1}{2}c$, then some complicated manipulation has to be done, and eight lines are required for determination of the sign. The proof in Art. 63 of Todhunter and Leathem begins with $\sin \tfrac{1}{2}(A+B)$, assumes no knowledge of the final result, is straightforward, and there is no question about the sign.

The formulae for the right-angled triangle should be derived, as mentioned in Art. 74, from the formulae for any triangle.

Ex. (i). Relation wanted between a, b, A where $C = 90°$.

Here a, C, b, A are four consecutive elements. Hence in the general triangle

$$\cos b \cos C = \cot a \sin b - \cot A \sin C;$$

but here $C = 90°$, and hence

$$0 = \cot a \sin b - \cot A,$$

or

$$\tan a = \tan A \sin b.$$

Ex. (ii). Relation wanted between A, B, c where $C = 90°$.
In the general triangle

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Thus when $C = 90°$, $\cos c = \cot A \cot B$.

I wish also to put in a protest against S in Note 1691. S is obtuse and to apply logarithms to formulae with $-\cos S$ is cumbrous. E, which comes in later in connection with the area of the triangle, should be used, where $A + B + C = 180° + 2E$, and then the last formula in Note 1691 would read

$$\cot R = \sqrt{\left\{\frac{\sin (A - E) \sin (B - E) \sin (C - E)}{\sin E}\right\}}.$$

All the quantities in the formula under the square root sign are positive.

LAWRENCE CRAWFORD.

"THE AUSTRALIAN MATHEMATICS TEACHER."

THIS new journal (Vol. I, No. 1, April 1945, pp. 24) owes its existence to the New South Wales Branch of the Mathematical Association, but the wider title is apt, since support and co-operation from teachers of mathematics in the other Australian States and in New Zealand have been promised. The journal is to promote "the exchange of ideas and experiences in the teaching of elementary mathematics" and "the instruction of teachers in trends and developments in mathematical education at home and abroad".

We can not refrain from remarking with pleasure that in size, style and content, the *Australian Mathematics Teacher*, with its Articles, Notes, Reviews and Gleanings, is closely akin to the *Gazette*; may this be a good omen for its long continuance.

A few copies of the first number are available, and can be had from T. A. A. Broadbent, Royal Naval College, Greenwich, London, S.E. 10, price 2s. 6d. post free.

WANTED.

Mathematical Gazette: any or all of Volumes I–VIII.

T. MARSDEN,
Fixby, 7 Perne Road, Cambridge.

1474. Mrs. H. M. McNab . . . said that the remark of a four years old boy led to the discovery of a new physics principle . . . which might lead to the repelling of the German robot bombs. . . . Mr. J. Scott Williams . . . said the principle of the device questioned Newton's third law of motion.—*Coventry Evening Telegram*, July 20, 1944. [Per Mr. C. C. Puckette.]

1475. THE ONE-EYED STANCE AGAIN.—It's going to be a critical year for Anglo-American relations. There will be 130,000,000 questioning eyes on us. —*Evening News*, Jan. 2, 1945. [Per Prof. L. M. Milne-Thomson.]

REVIEWS.

Lectures on the Theory of Functions. By J. E. LITTLEWOOD. Pp. 243. 17s. 6d. 1944. (Oxford)

For a mathematician who happens to be musical it is a common experience to be asked over and over again: how is it possible to combine two so divergent interests as mathematics and music? Professor Littlewood's book supplies a very powerful reply. I should say to a musical but unmathematical inquirer: study mathematics and read Littlewood's book on function theory, then you will see that mathematics, at its best, not only agrees well with music but simply *is* music. To follow up the way in which various themes are interlinked, statements are illuminated from every possible angle, different lines of approach are explored, to watch how every tool is made to yield the best possible result, induces an exalted state of mind closely akin to that experienced when listening to a great symphony or string quartet.

The subject-matter of the present volume I of the *Lectures on the Theory of Functions* is divided into three parts, an introduction, Chapter I and Chapter II. The introduction and Chapter I were printed in 1931, and in the form of offprints are familiar to many students who attended Professor Littlewood's lectures at Cambridge. The introduction contains among other material the bread and butter of the modern analyst, namely

Hölder's inequality: $|ab+cd| \leqslant (|a|^r+|c|^r)^{1/r}(|b|^{r'}+|d|^{r'})^{1/r'}$

$$\left(r>1,\ \frac{1}{r}+\frac{1}{r'}=1\right);$$

Minkowski's inequality:

$$|(a+b)^k+(c+d)^k|^{1/k} \leqslant (|a|^k+|c|^k)^{1/k}+(|b|^k+|d|^k)^{1/k} \quad (k \geqslant 1);$$

(and generalizations), the fundamental facts on Lebesgue integration and the whole apparatus of real variable technique centring round concepts such as

(i) interchanging of the operations of integration and passing to a limit,

(ii) comparison between various kinds of mean values of functions,

(iii) approximation to given functions by means of simpler functions belonging to specified classes.

The methods of proof are unified by the following three guiding principles:

(α) every (measurable) set is nearly a finite sum of intervals;

(β) every (integrable) function is nearly continuous;

(γ) every convergent sequence of functions is nearly uniformly convergent.

The introduction also contains sections on Fourier series and harmonic functions which present excellent systematic accounts of these two topics, the latter of which is apt to be neglected in England. Finally there are discussions of special "pathological" functions such as Weierstrass's continuous nowhere differentiable function $\sum_{n=0}^{\infty} a^{cn}z^{a^n}$, which are going to be used in a later volume for the purpose of showing that a number of results are best possible.

Chapter I takes up complex function theory in earnest. First there is a very thorough discussion of a group of classical theorems based on the maximum modulus principle: if $f(z)$ is regular at $z=z_0$, then $|f(z)|$ cannot attain a local maximum value at $z=z_0$ unless f is constant. This is followed by an admirable account of the fundamental problems of conformal representation leading up to Riemann's Theorem: given any two closed curves C_z, C_w it is possible to find a function $f(z)$, regular inside C_z, which has the

property that the equation $f(z)=w$ establishes a one-one correspondence between the interior D_z of C_z and the interior D_w of C_w. There is also a discussion of questions connected with the correspondence of boundary points of D_z, D_w, as well as a theorem on the connection between the mapping $w=f_n(z)$ of D_z on to one of a sequence of domains $D_w^{(n)}$ and the mapping

$$w=f^*(z) = \lim_{n\to\infty} f_n(z)$$

of D_z on to a region D_w^*, which might be written, in a certain sense, as $D_w^* = \lim_{n\to\infty} D_w^{(n)}$.

Many problems discussed in Chapter II are of the following type. If it is known that a function $f(z)$, regular in some region D, does not take anywhere in D certain given values w^*, how far does this fact prejudice the whole set of values actually taken by $f(z)$ in D? The most striking result in this direction is Picard's Theorem: if $f(z)$ is regular for all (finite) z, and if f never takes either of two values w_1, w_2, then f is a constant. Furthermore, this chapter contains the classical theorems associated with the names of Landau, Schottky and Bloch, as well as a fairly extensive discussion of so-called "schlicht" (simple) functions, *i.e.* functions which in a given region, say the unit circle, do not take any value more than once. There exist a number of different approaches to this group of theorems. It is an interesting feature of the present account that it makes consistent use of the modern idea of subordination (Lindelöf's principle) which was greatly developed by the author and W. W. Rogosinski—to whom the whole volume is dedicated—and which can be stated as follows. A function $f(z)$ is said to be subordinate to a function $F(z)$ if there is a function $\omega(z)$, regular in $|z|<1$ and satisfying $\omega(0)=0$,

$$|\omega(z)|<1 \text{ for } |z|<1,$$

such that

$$f(z)=f(\omega(z)) \text{ for } |z|<1.$$

by "Schwarz's Lemma"

$$|\omega(z)|\leqslant|z| \text{ for } |z|<1.$$

Hence f is obtained from f by applying a "shrinking-transformation" to the unit circle.

The study of such subordination relations is preceded by an account of the general theory of subharmonic functions, *i.e.* of real functions whose value at any point z_0 does not exceed their average value on any circle with centre z_0. Such functions are studied in this connection because some of the arguments on subordination are most easily understood in the light of this general concept of subharmonic functions.

A number of minor blemishes in the shape of misprints and little errors which are not included in the final section "Addenda and Corrigenda" cannot in the least detract from what constitutes a most important contribution to mathematical literature. In the opinion of the reviewer this importance is greatly enhanced by the fact that this book is most decidedly non-didactic—it contains not a single example—, that it does not compare the merits of different proofs of the same proposition on the basis of the number of lines required to write it down, that its sole aim seems to be to make the reader share the author's delight in one of the most beautiful realms of mathematics.

RICHARD RADO.

Algebra of Analysis. By K. MENGER. Pp. 50. $1. 1944. Notre Dame Mathematical Lectures, 3. (Notre Dame, Indiana)

In the development of the Calculus there are a number of steps which are purely algebraic in the sense that no limiting process is involved. For instance,

if the product rule $Duv = uDv + vDu$ has been established, and the derivative of x is known to be unity, then the derivative of $x^2 = x \,.\, x$ is given by

$$xDx + xDx = 2x.$$

Such steps form the subject-matter of what Menger calls the Algebra of Calculus. To take a slightly deeper example, the product rule also suffices to determine the derivative of $1/x$, since $0 = D\{x(1/x)\} = (1/x) \,.\, Dx + xD(1/x)$, and so $D(1/x) = -1/x^2$; of course this process *assumes* that $1/x$ is differentiable, and the Algebra of Calculus cannot prove this assumption.

Algebra of Analysis starts with an account of a new functional notation. Functions are denoted by single letters f, g, etc., and have no argument. Instead, we have three ways of combining functions, addition, multiplication and substitution, denoted by $f+g$, $f \,.\, g$ and fg, and representing what we ordinarily express by $f(x)+g(x)$, $f(x) \,.\, g(x)$ and $f(g(x))$ respectively. There are three special functions (called neutral elements) 0, 1, j which satisfy the postulates $f+0=f$, $f \,.\, 1 = f$, $fj = jf = f$, and $0 \neq 1 \neq j \neq 0$; furthermore, to each function f corresponds a function $-f$ such that $f+(-f)=0$. Menger gives no rules for the operation of the equality sign, but uses it in the usual way as a symmetrical transitive relation. Axioms are formulated making the three operations associative and distributive, and addition and multiplication commutative.

A function f is said to be constant if $f = f0$, and the constants are proved to form a ring which is closed with respect to substitutions. The fundamental assumption is then made that if $fc = gc$ for all constants c, then $f = g$. An algebra satisfying this assumption is said to have a base of constants.

Next the "lytic" functions "rec" and "neg" are defined by the equations $\text{rec} \,.\, j = 1$, $j + \text{neg} = 0$ respectively, so that rec is the reciprocal function and neg is $-j$. Since $(\text{rec} \,.\, j)\,0 = 10$ leads to $0 = 1$, the substitution of the function 0 in the function rec is forbidden (and so the axioms do not have universal application).

It is then assumed that there is an exponential function exp satisfying $\exp(g+h) = \exp g \,.\, \exp h$, $\exp \neq 0$, $\exp \neq 1$, and further that (when the ring of constants is real) this function admits an inverse function log satisfying $\log \exp = \exp \log = j$. Assuming further that there is a constant $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$, the function abs (*i.e.* $|x|$) is defined by $\text{abs} = \exp(\frac{1}{2} \,.\, \log(j \,.\, j))$, and the function sgn (*i.e.* the sign of x) by $\text{sgn} = \text{abs} \,.\, \text{rec}$, $\text{sgn}\,0 = 0$. The power function $c - \text{po}$ (*i.e.* x^c) is defined by $c - \text{po} = \exp(c \,.\, \log)$, and it is proved that $c - \text{po}$ and $\dfrac{1}{c} - \text{po}$ are inverse functions.

Circular functions are introduced by *assuming* that there is a function tan satisfying

$$\tan(g+h) = \frac{\tan g + \tan h}{1 - \tan g \,.\, \tan h}, \quad \tan \neq 0,$$

and a least $c > 0$ such that $\tan c = 0$.

From these assumptions it follows that $\tan c/2$ is undefined and that $\tan c/4$ has one of the values 1, -1.

sin and cos are defined by $\sin(2 \,.\, j) = \dfrac{2 \tan}{1 + \tan^2}$, $\cos(2 \,.\, j) = \dfrac{1 - \tan^2}{1 + \tan^2}$, and and $\sin/\cos = \tan$ is then derived.

Finally, tan is assumed to have an inverse arctan.

Looked at purely as an axiomatic system the algebra of functions suffers from the serious defect that so much has to be postulated which is readily proved in analysis. It would seem better to stress the value of the system as an auxiliary to analysis, rather than an alternative. The new functional

notation may also be of some importance in the philosophy of mathematics; the point that the familiar $f(x)$ may be regarded as the result of the substitution of the identity function x in the function f is well made.

The Algebra of Calculus forms the second chapter of the pamphlet. A differential operator D is postulated to satisfy the laws I, $D(f+g)=Df+Dg$; II, $D(f\,.\,g)=f\,.\,Dg+g\,.\,Df$; III, $D(fg)=Dfg\,.\,Dg$ (Dfg denotes $(Df)g$ throughout).

It is shown that $Dc=0$ when c is a constant, and that $Dj=1$, provided there are functions f for which $Df\neq 0$, and a number of simple derivatives are evaluated. For instance, D (rec g) = neg (-2) $-$ po $g\,.\,Dg$ and

$$D\exp = \exp D\exp 0.$$

In current notation the former is $D_x\{1/g(x)\}=-g'(x)/g(x)^2$ and the latter is $D_xE(x)=E'(0)E(x)$, where $E(x)$ is an exponential function. Menger is, of course, obliged to assume that rec g and exp are differentiable, and his proof is a transliteration of the following: $0=D\{g(1/g)\}=Dg\,.\,(1/g)+g\,.\,D(1/g)$, whence $D(1/g)=-Dg/g^2$; $D_xE(x+y)=E'(x+y)$ and

$$D_xE(x+y)=D_xE(x)E(y)=E'(x)E(y),$$

whence taking $x=0$, $E'(y)=E(y)E'(0)$.

The assumption is then made that there is an exponential function for which $D\exp 0=1$.

From the relation $\tan(j+c)=\dfrac{\tan j+\tan c}{1-\tan j\,.\,\tan c}$, the derivative of tan is obtained, provided $D\tan 0=1$ (the expression tan . tan c in § 5 is ambiguous and should read tan . (tan c)).

The indefinite integral (called antiderivative) is treated in a similar way. An equivalence relation $\sim$ is defined by the condition that $f\sim g$ if, and only if, $Df=Dg$, and an operator S is introduced to satisfy $D(Sf)=f$. It follows, writing Df for f, that $D(S(Df))=Df$, and therefore $S(Df)\sim f$.

The formulae for integrating by parts and by substitution are obtained, the latter in the neat form $Shg\sim S(hg\,.\,Dg)$.

It is perhaps surprising that S is used for the indefinite integral (suggesting a summation) rather than, say, I.

There seems to be little in the Algebra of Calculus to interest a student of modern analysis, but there may be some points of presentation which are of value to the teacher of elementary calculus.

The pamphlet concludes with a brief chapter on functions of several variables, and an account of the algebra of partial derivatives is promised in a future volume.

Some slight economy in hypothesis could be effected by formulating the system on the basis of the following three axioms.

I, $D(f+j)=Df+1$; II, $D(f\,.\,j)=f+j\,.\,Df$, III, $D(fg)=Dfg\,.\,Dg$.

From I and III we find

$$D[(F+j)g]=D(F+j)g\,.\,Dg=DFg\,.\,Dg+Dg;$$

write $Fg=f$ (*i.e.* $F=fg^{-1}$, where g^{-1} is the inverse of g), then

$$(F+j)g=Fg+jg=f+g, \quad \text{and} \quad DFg\,.\,Dg=Df,$$

and so:

$$(\alpha)\quad D(f+g)=Df+Dg.$$

Similarly from II and III,

$$D[(F\,.\,j)g]=D(F\,.\,j)g\,.\,Dg=(F+j\,.\,DF)g\,.\,Dg=Fg\,.\,Dg+g\,.\,DFg\,.\,Dg;$$

taking $Fg = f$, so that $(F \,.\, j)g = Fg \,.\, jg = f \,.\, g$, and therefore

$$(\beta) \quad D(f \,.\, g) = f \,.\, Dg + g \,.\, Df.$$

Moreover, from I, taking 0 for f, we have directly $Dj = 1$.

Alternatively, in place of II, we could take the axiom

$$\text{II}^* \quad D \exp = \exp,$$

by means of which (β) may be derived from I and III. For if log is the inverse of exp, so that $\exp \log = j$, then $\exp \log \,.\, D \log = 1$, and thus $j \,.\, D \log = 1$, whence $D \log = \text{rec}$. Therefore

$$D(f \,.\, g) = D[\exp(\log f + \log g)] = (f \,.\, g) \,.\, (\text{rec}\, f \,.\, Df + \text{rec}\, g \,.\, Dg)$$
$$= g \,.\, Df + f \,.\, Dg.$$

Errata. On p. 17, in the definition of even and odd functions, the terms even and odd should be interchanged. On p. 45, five lines from the bottom of the page, read h_{s_r} for s_r. R. L. G.

The Earth and the Sky. By A. ROBSON. Pp. 64. 2s. 6d. 1945. (Bell)

The descriptive side of Astronomy has long been popular, but in recent years the demands of Navigation have fostered an interest in the kinematics of the sky. This excellent little book bridges the gap between the popular and technical treatments of the subject, and the many definitions and new terms usually crowded into the first chapter of an advanced book are here amplified and explained, and their understanding assisted by the provision of many easy exercises. Enough spherical trigonometry is introduced, by way of the trigonometry of the "hillside" problem, to enable the simpler calculations of astro-navigation to be tackled.

The book will be well worth working through, not only by those whose careers may require it, but by any boy with a little time to spare who wishes to revise and extend his mathematics. The standard is not above that of the Additional Mathematics of the School Certificate, and the earlier chapters are well below this.

The diagrams could have been more interesting and self-explanatory, and the usefulness of Whitaker's *Almanack*—which is widely available—might have been mentioned. Quarter-squares and haversines are used but can be avoided. A. P. R.

A Note Book on Graphs. By R. J. GILLINGS. Pp. 32. 2s. 9d. 1945. (Edwards, Dunlop & Co., Sydney, N.S.W.)

This book from Australia is hand-written in printed capitals—war-time conditions made ordinary printing too expensive—but it is extremely clear and easy to read, and the diagrams are simple and well reproduced. It deals with graphs of the columnar type, straight line, locus, parabola, circle, ellipse, hyperbola, and there is a brief section describing more complex graphs such as the cycloid, catenary, sine curve and the spiral of Archimedes.

The aim of the book is to have a graph plotted easily and quickly without too much labour but with reasonable accuracy, and to lay stress on the interpretation of the graph and the results that may be obtained from it.

The approximate solution of quadratic equations by means of a parabola and a straight line is dealt with very fully. There are plenty of examples, with answers on the same pages as the questions.

Probably for economy of space, many facts about curves are stated which the pupil might have been led to discover for himself, but the skilful teacher should be able to draw these out from the pupil before he reads them in the book.

It is an attractive book, full of information. A class that has been taken through it should have acquired an excellent knowledge of graphs and of much algebra, and should have had plenty of interest aroused. A. W. S.

Formulaire pour le Calcul symbolique. Par N. W. McLachlan et P. Humbert. Pp. 65. 50 fr. 1941. Mémorial des Sciences mathématiques, 100. (Gauthier-Villars)

From one aspect, the operational calculus is the study of the relation between functions $f(t)$, $\phi(p)$ defined by

$$\phi(p) = p\int_0^\infty e^{-pt} f(t)\,dt.$$

Under reasonable restrictions on $f(t)$, it is easy to verify that the "functional image" of $f'(t)$ is $p\{\phi(p) - f(0)\}$, and that of $\int_0^t f(t)\,dt$ is $\phi(p)/p$. Thus limiting operations in one field correspond to algebraic operations in the other; and, for instance, if $f(t)$ satisfies a linear differential equation with constant coefficients, $\phi(p)$ satisfies an algebraic equation, in which the initial values of f and its derivatives are involved. We can infer properties of f from properties of ϕ, and the latter may be simpler to establish. To use the new language, we must know the fundamental grammatical rules and a skeleton vocabulary; but the need for a dictionary is obvious.

Just before the war, Dr. McLachlan, whose strong views on the importance of this calculus are well known to readers of the *Gazette*, had compiled an extensive dictionary; in collaboration with M. Pierre Humbert, this was prepared for the press, and publication followed in 1941. But, of course, it is only within the last few months that copies have become available in this country. The main sections are: a list of notations used for functions (for example, we must know precisely which definition of erf x is being used); the definitions and fundamental theorems of the calculus; the rules of manipulation (such as the two cited above); and the largest section, the list of functional images, in which Bessel functions are prominent. There is also a short section on the calculus with two variables, the correspondence being given by

$$\phi(p, q) = pq\int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\,dx\,dy.$$

Although several hundred images are listed, probably many more could be added now, as a result of work done during the war. But at the moment the list is the most extensive in print, to my knowledge, and so should be at the elbow of anyone who uses this calculus extensively.

The typography is of the standard which we expect from MM. Gauthier-Villars; and it would be improper to close this review without an expression of pleasure at seeing once again the well-known imprint and of hope that this will be but the first of a flow of mathematical treatises to reach this journal from a firm whose productions figure so largely on the shelves of every mathematician. T. A. A. B.

1476. THE MIDDLE ANGLES.—The king reigning over that part of the country now called Lincolnshire and Cambridge in A.D. 609 was Eghedde the Bald. The people of his kingdom were Middle Angles—that is, they were neither right nor straight—and were known as the Treacherous Forty-Fives. —J. N. Chance, *Murder in Oils*, Chap. I (1945). [Per Prof. E. H. Neville.]

LONDON BRANCH.

On 13th October, 1945, Mr. H. A. Baxter of Liverpool Institute High School spoke on "The Teaching of Mechanics to the Average Pupil". He outlined a course of mechanics which could be taken in place of algebra and geometry by pupils weak in these subjects. Emphasis was laid on the fundamental ideas of time, motion and mass, these being studied before the introduction of force, which was to be defined in terms of mass and acceleration. In this way statics was postponed until considerable work on motion had been done.

On 10th November, 1945, Professor Roberts gave his postponed Presidential Address on "Mathematics at the R.M.A., Woolwich, in the eighteenth and nineteenth centuries—a talk on forgotten worthies".

A. J. Taylor, *Hon. Secretary.*

MANCHESTER BRANCH.

Report for 1944–5.

On 16th May, 1944, Mr. Cornish, of the Manchester College of Technology, gave a paper entitled "Approximate solutions of equations; some methods used by engineers". This meeting was very well attended, and an interesting paper was made clearer by the provision of printed examples of the methods described.

The meetings on 26th September and 14th November were devoted to discussion of a syllabus of school work in mathematics given by three members, and it was decided to set up a sub-committee to formulate a common syllabus for the first two years in secondary schools of all types. This sub-committee has met many times and has now completed its work and hopes to report to the Branch in October.

On 6th February, 1945, Dr. J. C. P. Miller of Liverpool University gave a paper on "Archimedean Polyhedra", with sixty beautifully constructed models.

The meeting arranged for 8th May, 1945, to have been addressed by Mr. Edgar, H.M.I., was re-arranged for September.

The officers for the session were: *President*, Miss E. M. Holman; *Treasurer*, Miss Garner; *Secretary*, Mr. G. D. Pope. G. D. Pope, *Hon. Secretary.*

MIDLAND BRANCH.

The Midland Branch, which draws members from Warwickshire, Staffordshire, Shropshire and Worcestershire, has now a membership of 94. During the session 1944–5, four meetings were held at the University, Birmingham.

Autumn Term, 1944. On 4th November Mr. Caton gave his Presidential Address, published in full in the *Gazette* for July 1945. Forty members were present.

On 2nd December Mr. Waterhouse, of the Biology Department of the University of Birmingham, gave a paper on "Mathematical Genetics" to eighty members. Starting from Mendel's work on pairs of hereditary factors, Mr. Waterhouse gave a lucid account of some of the mathematical relations involved in the modern theories concerning dominant and recessive genes, and illustrated these relations with examples from the plant and animal worlds.

Spring Term, 1945. On 3rd February the Annual General Meeting was held. Seventy members were present. After the Treasurer had presented his report, which showed that the receipts for the year ending 31st October, 1944, had exceeded the expenditure by £5 8s. 9d., the officers for the session 1945–6

were elected as follows: *President*, Mr. E. V. Smith; *Hon. Secretaries*, Miss L. E. Hardcastle, Mr. C. T. Lear Caton; *Treasurer*, Mr. M. A. Porter; *Committee*, Misses Boyter, Cook, Le Quesne and Tebbutt, Dr. Pedoe, Messrs. Cooper, Sheen and Wardle. The Business Meeting was followed by the reading of three papers: Mr. Hughes, Headmaster of Crockett's Lane Junior School, Smethwick, on "Arithmetic in the Junior Schools, particularly the work in the year preceding entrance to the Secondary Schools"; Mr. Richards, of Oldbury Secondary School, on "Mathematics in the First Year of the Secondary School"; Mr. Porter, of King Edward's High School, Birmingham, on the same topic as Mr. Richards, but chiefly from the point of view of a High School in a large town.

On 3rd March, Mr. W. W. Sawyer, of Leicester College of Technology, gave a paper on "The Use of History in the Teaching of Mathematics". Taking as his theme that practical life forces us to make discoveries, Mr. Sawyer began by mentioning some lines of attack by which problems, now needing the use of calculus and other advanced mathematical methods, were solved in a simple way some centuries ago before these mathematical methods were invented, and stated his view that such simple lines of attack may form a helpful approach in present-day mathematical teaching.

L. E. HARDCASTLE, C. T. L. CATON,
Joint Hon. Secretaries.

QUEENSLAND BRANCH.

REPORT FOR THE YEAR 1944–5.

THE Annual Meeting was held at the University on Friday, 11th May, 1945. The Presidential Address by Professor Simonds was on the subject "James Stirling".

During the year two General Meetings were held at the University. At the first, on 28th July, 1944, Mr. J. P. McCarthy read a paper on "The Cardioid", and at the second, on 3rd November, 1944, Mr. P. B. McGovern read a paper on "Some Applications of Statistics".

The number of members is 8, the number of associate members is 22; four are on duty with the Services. During the year the death occurred of Mr. S. Stephenson, who was a foundation member and took an active part in the activities of the Branch.

The statement of receipts and expenses showed a credit balance of £13 5s. 2d. Attendance at meetings has been maintained, and copies of the *Mathematical Gazette* are circulated amongst associate members.

The Committee for the coming year is: *President*, Professor E. F. Simonds; *Vice-Presidents*, Mr. R. A. Ker, Mr. I. Waddle; *Hon. Secretary and Treasurer*, Mr. J. P. McCarthy; *Members*, Miss E. H. Raybould, Mr. E. W. Jones, Mr. S. G. Brown, Mr. P. B. McGovern. J. P. MCCARTHY, *Hon. Secretary.*

SOUTHAMPTON BRANCH.

THE Southampton Branch has recommenced its activities. The first meeting since the war was held at University College, Southampton, on 26th October, 1945. There was a good attendance and Dr. F. G. Maunsell described some of his mathematical experiences during his service in the Army. Professor Ruse was elected President and Dr. Maunsell, University College, Southampton, secretary and treasurer for the session. It is hoped that all members of the Association who live in the district will get into touch with the secretary and, if possible, attend the meetings. F. G. MAUNSELL, *Hon. Secretary.*

VICTORIA BRANCH.

THE Branch resumed regular meetings in 1944, these having been suspended since 1940, while active members were largely engaged in work as R.A.A.F. Hon. Instructors (*Gazette*, XXVI, 269). The President (since 1937), Mr. D. K. Picken, officers and committee were re-elected: Mr. H. B. Sarjeant as Acting Secretary in the absence of the Hon. Secretary on active service (see below). There are 8 members and 20 associates of the Branch.

Five meetings were held, in March, April, July, August and October: the speakers were Professor Cherry ("New Matriculation Mathematics"); Dr. A. R. Miller, a Melbourne and Cambridge graduate, on loan to A.O.C., R.A.A.F. Command, from the British Air Ministry, as Scientific Adviser ("Air Warfare Research"); Mr. J. M. Allen, Head of the Mathematics Department of the Melbourne Technical College ("Dimensional Analysis and the π-theorem"); Mr. T. A. Lang, Engineer of the State Rivers and Waters Commission, assisted by Mr. J. D. Lang with film strip projector ("Graphical Methods"); Mr. E. H. Palfreyman, of P.M.G.'s Research Laboratory ("Calculating Machines", illustrated by a large number of machines, of a variety of types and values).

The Hon. Secretary of the Branch (S/L. F. J. D. Syer), who is on active service with the R.A.A.F. as a scientific observer, was awarded the decoration of Member of the Order of the British Empire. The conclusion of the citation is as follows: "By his courage and devotion to duty F/L. Syer rendered inestimable service to the R.A.A.F." A minute of appreciation and congratulation was put on record, and a copy forwarded.

This Branch has given cordial support and co-operation to the New South Wales Branch in the promotion of a journal of elementary mathematics for Australia and New Zealand, to be published in 1945.

During the year opportunity was taken for some Press propaganda on Mathematics as Culture (a subject on which the postponed Presidential Address of the retiring President will be delivered in 1945).

D. K. PICKEN, *President.*
H. B. SARJEANT, *Acting Secretary.*

BOOKS RECEIVED FOR REVIEW.

R. J. Gillings. *A Notebook on graphs. Columnar, Straight Line, Locus, Curved and Complex.* Pp. 32. 2s. 9d. 1945. (Dunlop, Sydney, N.S.W.)

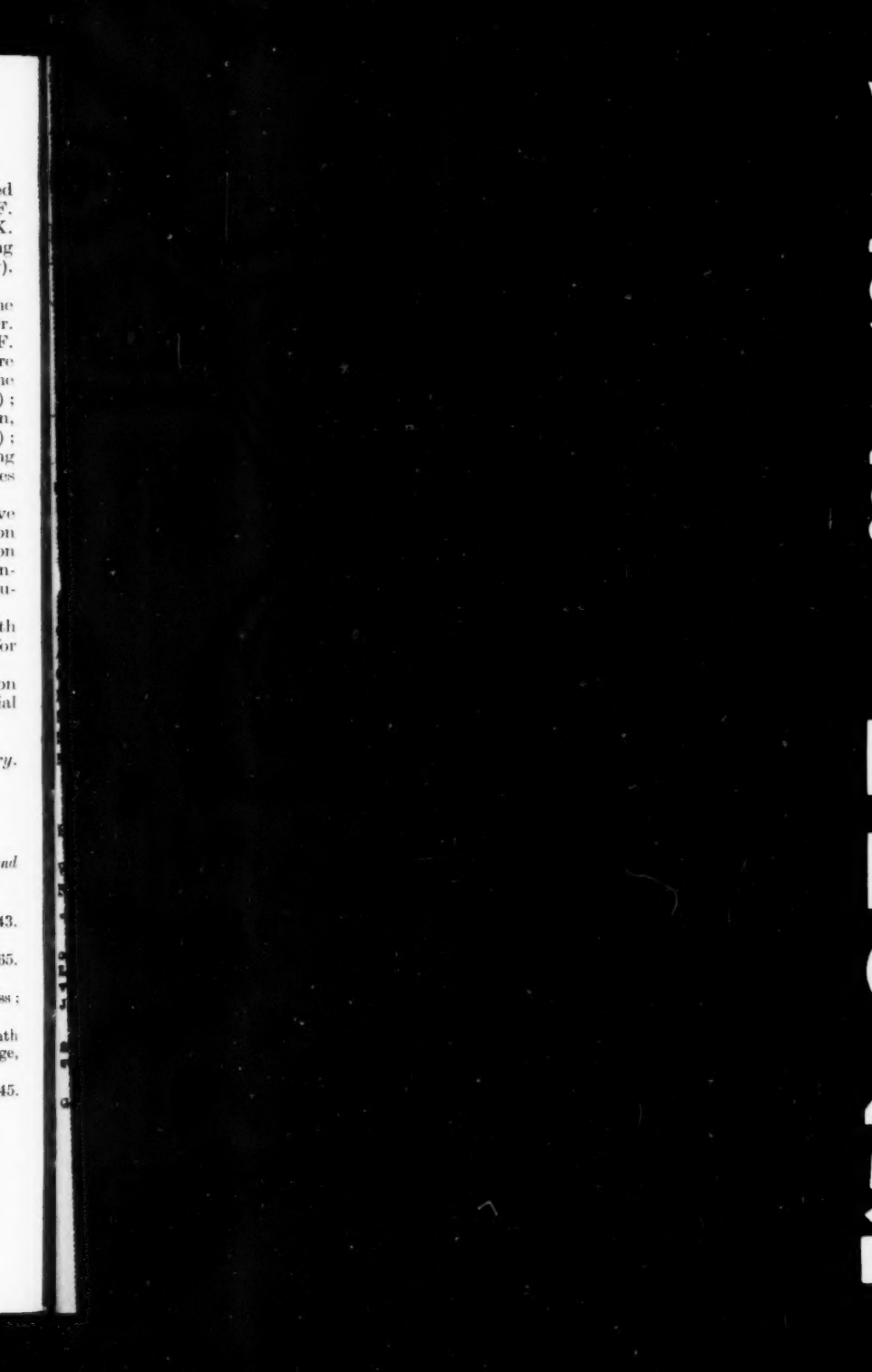
S. H. Glenister. *Builders' calculations.* Pp. 125. 5s. 1945. (Harrap)

J. Hadamard. *The psychology of invention in the mathematical field.* Pp. xiii, 143. 13s. 6d. 1945. (Princeton University Press; Humphrey Milford)

N. W. McLachlan et P. Humbert. *Formulaire pour le Calcul symbolique.* Pp. 65. 50 fr. 1941. Mémorial des Sciences mathématiques, 100. (Gauthier-Villars)

G. Polya. *How to solve it.* Pp. xv, 204. 16s. 6d. 1945. (Princeton University Press; Humphrey Milford)

Multi-sensory aids in the teaching of mathematics. Pp. xv, 455. $2. 1945. Eighteenth Yearbook of the National Council of Teachers of Mathematics. (Teachers' College, Columbia University, New York)

Science in progress. 4th series, edited by G. A. Baitsell. Pp. xvi, 331. 20s. 1945. (Yale University Press; Humphrey Milford)

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